Remarkable coincidence for the top Yukawa coupling and an approximately massless bound state

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Abstract

We calculate, with several corrections, the nonrelativistic binding by Higgs exchange and gluon exchange between six top and six antitop quarks (actually replaced by left-handed b quarks from time to time). The *remarkable result* is that, within our calculational accuracy of the order of 14% in the top-quark Yukawa coupling g_t, the experimental running top-quark Yukawa coupling $g_t = 0.935$ happens to have just that value which gives a perfect cancellation of the unbound mass = 12 top-quark masses by this binding energy. In other words the bound state is massless to the accuracy of our calculation. Our calculation is in disagreement with a similar calculation by Kuchiev et al., but this deviation may be explained by a phase transition. We and Kuchiev et al. compute on different sides of this phase transition.

1   Introduction

We have earlier claimed [1] that, if of the order of 6 top quarks and 6 antitop quarks were brought within the distance of the Higgs Compton wavelength from each other, they would obtain such a strong binding energy that it would be definitely of *the same order of magnitude as the mass energy of these 6 top and 6 antitop quarks*. Within the accuracy of the previous calculations [1, 2, 3], it was not excluded that the binding energy for the 6 top plus 6 antitop bound state could just compensate the mass energy, so that the total mass of the bound state would be just zero. Indeed we concluded that, within uncertainty, we had consistency with the hypothesis that the experimental top-quark Yukawa coupling constant g_t had been "mysteriously" tuned so as to make this bound state of 6 top and 6 antitop quarks—called NBS (new bound state) or the t ball—have just zero mass. In our notation (see Appendix A), the experimental top-quark Yukawa coupling is 0.935 corresponding to the running mass or 0.992 if we use the pole mass of 172.6 GeV [4].

However, it was recently claimed by Kuchiev, Flambaum and Shuryak [5] that, with the experimental coupling $g_t \approx 0.989$ and a realistic Higgs mass used to give the Yukawa form of the potential, the system of the 6 top and 6 antitop quarks would not even bind, let alone bind to zero mass. As we shall show in Appendix J, the calculation of the bound state mass or rather the lowest energy for the $6t + 6\overline{t}$ system as a function of the Yukawa coupling g_t has the character of a "phase transition." That is to say that the mass m_{bound} of the bound state gets a kink as a function of the Yukawa coupling g_t. In this Appendix we shall only present a toy model calculation showing such a kink, but the expectation is that also the fully correct calculation would at least approximately show a kink. Having such a nonanalytic, or at least essentially nonanalytic, behavior of the bound state mass in mind, it could easily happen that the fact that Kuchiev et al. ignore some corrections could lead them to a qualitatively wrong conclusion by working on the wrong side of the phase transitionlike kink. As we illustrate with our toy model in Appendix J, the suggested phase transition should be caused by the vacuum Higgs field collapsing under the influence of the high density of top and antitop quarks in the potential bound state.

Two calculations were made in Ref. [5] using, respectively, a variational method and self-consistent Hartree Fock equations. As we remark in Appendix I our own estimate of what we call the many body effect agrees with the results of this reference in the massless Higgs approximation. Kuchiev et al. effectively included u-channel Higgs exchange and also gluon exchange, in addition to the explicitly considered t-channel Higgs exchange, by increasing the t-channel Higgs exchange potential by a factor of 2. However, in the present article we want to take into account some effects not considered in Ref. [5]. The two most important of these are as follows:

1) First, one has to take into account the possibility that, *if the system of the 6 top plus 6 antitop quarks indeed binds strongly*, then inside the bound state the Higgs field can be strongly reduced compared to its usual vacuum expectation value (VEV). Since the second derivative of the effective potential for the Higgs field can even be negative, leading to an effectively tachyonic Higgs particle, for small values of the Higgs field, it follows that the effective Higgs mass inside the bound state might be considerably smaller than the usual Higgs mass outside. This correction of using an effective Higgs mass only sets in when indeed one has the bound state. So it is *a priori* not excluded that there could be a binding due partly to this effect, while a calculation not using an effective Higgs mass might still show no binding.

2) We shall take into account also the exchange of W and Z bosons and even the photon. What really matter most in this connection are, as we shall see below, the components of the intermediate gauge bosons which in reality correspond to the eaten Higgs field components. So it is really the exchange of components of the Higgs field, other than the radial one identified with the genuine Higgs particles, which we consider here.

In addition we shall present several smaller corrections which were not considered in Ref. [5].

In the present article we correct and improve our previous calculations [1, 2, 3] of the "critical" top Yukawa coupling $g_t|_{phase\ transition}$ needed to make the bound state, NBS or t ball or dodecaquark, massless. As mentioned above we include first of all part of the effect on the binding coming from the exchange of W and Z bosons, namely, the part that is in reality the exchange of the "eaten Higgses." Even in the limit of vanishing fine structure constants, α_1 for $U(1)$ and α_2 for $SU(2)$, respectively, there would be an exchange interaction between bottom and/or top quarks by the exchange of W's or Z's. This is because the squared masses in the propagators of the gauge bosons are proportional to the fine structure constants and consequently, for part of the exchange potential, the fine structure constants drop out of the calculation. It is a part we interpret as being really the exchange of an eaten Higgs component. Inclusion of at least part of the W exchange means that a top quark gets converted to a b quark or oppositely. Including such contributions, we thus have to imagine that our NBS or t-ball bound state is a superposition not only of top and antitop quarks, but also has components with some of these top or antitop quarks replaced by b quarks or anti-b quarks, respectively. It is, however, trivial to see that the right-handed chirality b quark is totally decoupled in the first approximation and cannot come into this approximation.

We consider the inclusion of this weak interaction exchange so to speak—really of eaten Higgses—as a major correction, on top of which we shall then further make a

series of "smaller" corrections, which are typically really not so much smaller. The whole calculation is basically a nonrelativistic one, although one of our many corrections is an attempt to take into account in a somewhat crude way relativistic effects.

The major purpose of these calculations is to see to what accuracy we should be able to claim that the cancellation of the binding energy and the energy of the masses of the constituents occur just for the experimental top-quark Yukawa coupling. If we could claim the accuracy is high enough, it would mean that nature had chosen a very special value for the top-quark Yukawa coupling, and there would be some mysterious fine-tuning problem to be explained—for instance by our multiple point principle [1, 6].

But even not worrying about the strange coincidence which such a special top Yukawa coupling would mean, it would say that now the bound state became of such low mass that there might be a hope of producing it. In fact we expect a spectrum of bound states with different numbers of top-quark constituents, as discussed in Appendix H, and some of these should be found [7] at the LHC and conceivably could already have been produced at the Tevatron.

It should be emphasized that we *only use the standard model* to calculate the critical Yukawa coupling needed to make the bound state mass zero. If indeed the binding is small (or it does not bind at all as Kuchiev et al. [5] would claim), even after additional corrections the system will remain nonrelativistic. However, if the binding gets of the same order as the mass energy ($12m_t$), relativistic considerations are called for. Kuchiev et al. never need such relativistic corrections; but we formulate our whole calculation the opposite way around, in as far as we formulate it as calculating that specific value of the top-quark Yukawa coupling g_t, which makes the bound state of the 6 top and the 6 antitop quarks massless. Doing it this way immediately forces us into the consideration of relativistic calculations.

Our attempts to make the best relativistic estimate for instance lead us to think of our calculation being done in the infinite momentum frame (see Sec. 2 and Appendix B). In this frame we find that the binding needed to make the bound state mass zero is decreased by a factor of 2 compared to what a naive nonrelativistic calculation would suggest. It corresponds to extrapolating the mass squared linearly as a function of the binding energy rather than extrapolating the mass. Also the relativistic speeds mean that the exchanged Higgses cannot necessarily be fully described by means of static Coulomb forces, but that ladder diagrams with crossing rungs (lines) have to be included (see Sec. 8).

In Sec. 2 we shall shortly review and update the nonrelativistic calculation, formulated as a calculation of the value of the top-quark Yukawa coupling g_t needed for the cancellation of the binding energy with the constituent masses. In Sec. 3 we shall make

the first crude introduction of the important effect of including the eaten Higgs exchanges and the thereby associated introduction of a component of left-handed b quarks in the bound state.

In the following sections we shall then go through several "smaller" corrections to our calculation: In Sec. 4 we make some correction to our too crude treatment of the eaten Higgs exchange force.

In Sec. 5 we discuss the correction of the Higgs mass after all not being exactly zero, so that in principle we have a Yukawa potential rather than, as we used at first, a massless Higgs approximation meaning a Coulomb-like potential. However the point will be that, due to the Higgs field from the many top quarks and antitop quarks largely compensating or even overcompensating the vacuum expectation value of the Higgs field, the effective potential for the Higgs field has a second derivative (corresponding to the effective mass squared) in the relevant region which is even some times negative. It follows that an effective Higgs mass, introduced to approximately describe the relevant situation, will be much smaller than the physical Higgs mass. For definiteness, in this paper we shall take the physical Higgs mass to be $m_h = 115$ GeV, corresponding to the LEP lower bound which coincides with the 2 standard deviation hint of a Higgs signal [8]; this value is also consistent with our multiple point principle prediction [7, 9, 10]. However we shall also take a conservative error of ± 50 GeV on m_h.

In Sec. 6 we include the s-channel exchange of both the Higgs and gluons; these contributions were left out in our previous calculations, since they are more difficult to estimate and more uncertain.

Then in Sec. 7 we consider the correction due to the dependence of the top-quark mass on the value of the Higgs field inside the bound state. A correction due to the finite speed of Higgs exchange is considered in Sec. 8.

In Sec. 9 we correct for the very crude way in which we previously treated the genuine many body problem, which occurs when we have a system of 12 constituent particles. Previously we assumed that we could calculate this effect by letting each top or anti-top quark "feel" the field of 11/2 of the other 11 particles, meaning that there is on the average 11/2 other particles inside a sphere around the center of mass point reaching out to the particle in question and 11/2 outside. The field from the outside ones is supposed to be negligible on the average, while that from the inside ones can be treated as if they were all in a "nucleus" in the center. As mentioned above this many body effect was also calculated in Ref. [5]. Our results for the size of the effect are in agreement with this paper for the massless Higgs exchange approximation.

In Sec. 10 we consider the contribution to the binding energy from the exchange of $SU(2)$ gauge bosons. Then in Sec. 11 we consider $U(1)$-gauge boson exchange.

In Sec. 12 we discuss at what precise value of the scale μ we think that our calculation delivers the top-quark running Yukawa coupling $g_t(\mu)$. We call this the renormalization group correction.

Then in Sec. 13 we present the final results and collect an estimated uncertainty for the various corrections and thereby essentially for the whole calculation.

In Sec. 14 we present the conclusion and discussion.

2   Nonrelativistic binding of top and antitop quarks by Higgs and gluon exchanges

Calculating bound states in relativity is generally difficult and in principle should be done using the Bethe-Salpeter equation. However, it is much easier to work with atomic physics, e.g., the Bohr atom, and the infinite momentum frame technology, which to some extent has similar simplifying properties to the nonrelativistic approximation, and this is what we in principle shall use now. We shall use first the nonrelativistic approximation and claim that, if we only consider the t-channel exchange and only one constituent going around a central particle or bunch of particles, we can simply use the old Bohr formulas for the hydrogen atom if we ignore the mass of the exchanged Higgs particle being different from zero—i.e., we use a Coulomb potential rather than the true Yukawa one. But then we argue that, for weak binding, we can trivially derive the binding in the infinite momentum frame from the nonrelativistic one, so that we can essentially use the Bohr formula also in the infinite momentum frame. It is our intention next to include also u-channel exchange while leaving the s-channel exchange, which gets appreciably more complicated, to the later Sec. 6.

The virtual exchange of the Higgs particle between two quarks, two antiquarks or a quark-antiquark pair yields an attractive force in each case. For top quarks Higgs exchange provides a strong force, since we know phenomenologically that $g_t(\mu) \sim 1$ in a notation in which the Lagrangian density for the Higgs top-quark interaction is $\frac{g_t}{\sqrt{2}}\overline{\psi}_{tL}\phi_h\psi_{tR} + h.c.$, where then the Higgs field is normalized to the expectation value $< \phi_h >= 246$ GeV. See Appendix A for our notation. In this notation the potential between two top or antitop quarks, using only the t-channel exchange with massless Higgs particles, is

$$V_{t-channel\ Higgs} = -\frac{g_t^2/2}{4\pi r}. \tag{1}$$

So let us now consider putting more and more t and $\bar{t}$ quarks together in the lowest energy relative s-wave states, the 1s wave. The Higgs exchange binding energy for the whole system becomes proportional to the number of pairs of constituents, rather than to

the number of constituents. So *a priori*, by combining sufficiently many constituents, the total binding energy could exceed the constituent mass of the system. However we can put a maximum of $6t + 6\bar{t}$ quarks into the ground state 1s wave. We shall now estimate the binding energy of such a 12 particle bound state.

As a first step we consider the binding energy $-E_1$ of one of them to the remaining 11 constituents treated as just one particle, analogous to the nucleus in the hydrogen atom but consisting of $Z = 11$ quarks.

However, if we want to be allowed to sum the various E_1s obtained for the 12 constituents, in order to obtain the total potential energy of the system (as we must to calculate the bound state mass), we must think of bringing the quarks or antiquarks into the bound state one by one. That is to say that, when we bring in the i'th constituent, the number of constituents in the center is only $i - 1$, so that the potential is $-\frac{(i-1)g_t^2/2}{4\pi r}$. So, instead of taking the potential felt by a single constituent in the final situation (i.e., in the bound state) $V = -\frac{11g_t^2/2}{4\pi r}$, we take an average over the steps of putting in the particles one by one and use the potential:

$$V = -\frac{\frac{11}{2}g_t^2/2}{4\pi r}. \tag{2}$$

We assume that the radius of the system turns out to be reasonably small, compared to the Compton wavelength of the Higgs particle, and use the well-known Bohr formula for the ground state energy level of a one-electron atom with atomic number $Z = 11$, but modified by the just mentioned inclusion of a factor $1/2$ *in the potential* to obtain the crude estimate[1]:

$$E_1 = -\left(\frac{\frac{11}{2}g_t^2/2}{4\pi}\right)^2 \frac{11m_t}{24}. \tag{3}$$

Here g_t is the top-quark Yukawa coupling constant, in our normalization in which the top-quark mass is given by $m_t = g_t\,174$ GeV. Furthermore we used the reduced mass of $\frac{11}{12}m_t$ for the one top quark moving relative to the other 11.

The nonrelativistic binding energy of the 12 particle system is then given by $E_{binding} = -12E_1$. This estimate only takes account of the t-channel exchange of a Higgs particle between the constituents.

2.1   u channel

A simple estimate of the u-channel Higgs exchange contribution [1] increases the binding energy by a further factor of $(16/11)^2$. This can be seen as follows. Considering that the

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[1]This formula actually represents a correction by a factor of 2 compared to our previous publications [1, 2, 3] in which we instead divided the *binding energy* computed for $Z = 11$ by 2; but then one has forgotten that the factor $1/2$ in the potential ends up being *squared* in the binding energy (the Rydberg in Bohr's formula). This is the effect of the average radius increasing when the potential is decreased.

system is totally antisymmetric in spin and color permutations, we can effectively proceed as if it consisted of 6 top quarks and 6 antitop quarks, with both of these bunches being bosons. Then the permutation of the interacting particles caused by going from t-channel to u-channel exchange means adding them up as if the force were twice as big. Since the considered quark can be permuted in this way with the remaining 5 quarks out of the other 11 quarks or antiquarks, we conclude that the factor of 11 inside the square in Eq. (3) should be replaced by $11 + 5 = 16$. This gives:

$$E_{binding} = -12 * (16/11)^2 E_1 = \frac{11g_t^4}{2\pi^2} m_t = 0.557 m_t g_t^4. \tag{4}$$

Inclusion of the u-channel contribution in this way is equivalent to using an averaged potential of

$$V_{with \ u-ch} = -\frac{\frac{16}{2}g_t^2/2}{4\pi r}. \tag{5}$$

2.2 Gluon exchange

We have so far neglected the attraction due to the exchange of gauge particles. So let us estimate the main effect coming from gluon exchange [2] due to the interaction $g_s \overline{\psi_t} A_\mu^a \lambda^a / 2 \gamma^\mu \psi_t$. It follows that the t-channel gluon exchange graph gives an effective Coulomb potential for a quark-antiquark pair in a color singlet state of

$$V_{gluon} = -\frac{g_s^2 Tr(\lambda^a/2 * \lambda^a/2)_{\underline{3}}}{4\pi Tr(I)_{\underline{3}} r} = -\frac{g_s^2 8/2}{4\pi * 3r} = -e_{t\bar{t}}^2/(4\pi r). \tag{6}$$

The QCD fine structure constant is given by $\alpha_s(M_Z) = g_s^2(M_Z)/4\pi = 0.118$. However, as will be discussed in Appendix C, the scale associated with the radius of the new bound state is closer to the m_t scale than to the M_Z scale. We will therefore take the value $\alpha_S(m_t) = 0.109$ in our estimate. This corresponds to an effective gluon $t - \bar{t}$ coupling constant squared of

$$e_{t\bar{t}}^2 = \frac{4}{3}g_s^2 = \frac{4}{3}1.37 = 1.83. \tag{7}$$

Here, however, we must bear in mind that the gluon exchange potential (6) is only for a quark attracting an antiquark in the compensating color state. It is not the coupling between all pairs of quarks and antiquarks; rather we should consider an averaged gluon potential as follows.

For definiteness, consider a t quark in the bound state; it interacts with 6 $\bar{t}$ quarks and 5 t quarks. The 6 $\bar{t}$ quarks form a color singlet and so their combined interaction with the considered t quark vanishes. On the other hand, the 5 t quarks combine to form a color antitriplet, which together interact like a $\bar{t}$ quark with the considered t quark. So the total gluon interaction of the considered t quark is the same as it would have been

with a single $\bar{t}$ quark. In this case the u-channel gluon contribution should equal that of the t channel. We shall also include a factor $1/2$, analogous to that included in the Higgs potential V above, which takes into account the probability of the center of the effective 5 quark system being closer to the center of the bound state than the considered quark. The averaged gluon potential to be used, in analogy to the t plus u channel Higgs exchange potential $V_{with\ u-ch}$ of (5), thus becomes 2 times $1/2$ of the expression (6), i.e., accidentally just V_{gluon} itself. Thus the full averaged potential, to be used as if all the quarks and antiquarks interacted in the same way, is

$$V_{total} = V_{gluon} + V_{with\ u-ch} = -\frac{e_{t\bar{t}}^2}{4\pi r} - \frac{\frac{16}{2}g_t^2/2}{4\pi r} = -\frac{e_{t\bar{t}}^2 + 4g_t^2}{4\pi r}. \tag{8}$$

This means that the binding energy (4) should be corrected to include the gluon exchange force by substituting

$$4g_t^2 \rightarrow e_{t\bar{t}}^2 + 4g_t^2, \tag{9}$$

which leads to (4) being replaced by

$$
\begin{aligned}
E_{binding} &= \left(\frac{11(e_{t\bar{t}}^2 + 4g_t^2)^2}{32\pi^2}\right) m_t & (10)\\
&= 0.0348 m_t (e_{t\bar{t}}^2 + 4g_t^2)^2 & (11)\\
&= 0.557 m_t (0.456 + g_t^2)^2. & (12)
\end{aligned}
$$

Later on, as we see that both the experimental g_t value and the critical g_t value (which we are about to estimate) are close to unity, it follows that as far as the coupling squared is concerned the gluon potential is about half as strong as the Higgs potential.

2.3 Infinite momentum frame

We can always think of our system as moving with a specified high momentum in the z direction. This is really considering the infinite momentum frame. As long as the binding is so small that higher order in it is irrelevant, we can trust the nonrelivistic approximation and even translate it into infinite momentum frame (IMF) language, in which the energy E_{IMF} of a system of mass m having large momentum component p_z and transverse momentum $\vec{p}_T$ is generally written in the form

$$E_{IMF} = p_z + (m^2 + \vec{p}_T^2)/(2p_z). \tag{13}$$

When we have an object composed of several particles, each of them must have its large momentum component $p_{zi} = x_i p_z$, where then p_z stands for the total momentum of the cluster of particles in the z direction. In this notation the total infinite momentum frame

energy for such a cluster of n constituent particles becomes

$$E_{IMF\ cluster} = p_z + \left(\sum_{i=1}^{n} (m_i^2 + \vec{p}_{Ti}^2)/x_i \right) /(2p_z) + \text{interaction terms.} \tag{14}$$

The reason we propose to think about this infinite momentum frame—without even doing any proper calculation in it—is that in this language we keep to the nonrelativistically looking formula,[2] as long as p_z is very large, even if the mass squared m_{bound}^2 of the bound state[3] we wish to study should become small. Then we can, namely, imagine that one can calculate the energy $E_{IMF\ bound}$ of the bound state, due to the Higgs and gluon exchange, in the nonrelativistic way even when the mass squared m_{bound}^2 is close to zero. Supposing that this can be done in a formalism in which one has a Hamiltonian involving the x_i's and their conjugates, as well as the transverse momenta and their conjugates, we should expect the $E_{IMF\ bound}$-energy eigenvalue to be analytic as a function of the parameters. Hence m_{bound}^2, which is linearly related to this energy, should also be analytic. Ignoring in such a calculation higher order terms in the coupling g_t than the fourth order, which we just used, we can then reliably get the mass squared of the bound state m_{bound}^2 even become negative, provided that the interaction is sufficiently strong.

That is to say that we can now obtain a tachyonic bound state with $m_{bound}^2 < 0$. In this way a new vacuum phase could appear due to Bose-Einstein condensation. Let us consider a Taylor expansion in g_t^2 for the mass *squared* of the bound state, estimated from our nonrelativistic binding energy formula:

$$\begin{align}
m_{bound}^2 &= (12m_t)^2 - 2\,(12m_t) \times E_{binding} + ... \tag{15} \\
&= (12m_t)^2 \left(1 - \frac{2 * 0.557(0.456 + g_t^2)^2}{12} + ... \right) \tag{16} \\
&= (12m_t)^2 \left(1 - 0.0929 * (0.456 + g_t^2)^2 + ... \right). \tag{17}
\end{align}$$

Assuming that this expansion can, to first approximation, be trusted—as our argument using the infinite momentum frame was meant to suggest—for large g_t, the condition $m_{bound}^2 = 0$ for the appearance of the above phase transition with degenerate vacua becomes to leading order[4]:

$$0 = 1 - 0.0929 * (0.456 + g_t^2)^2 \tag{18}$$

or

$$g_t|_{phase\ transition} \simeq \sqrt{\sqrt{1/0.0929} - 0.456} = 1.68. \tag{19}$$

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[2]It is non-relativistic in the sense that the kinetic term is quadratic in $\vec{p}_T$ [11].

[3]The mass squared m_{bound}^2 of the bound state is defined such that, if the constituent wave function corresponding to the bound state NBS is used for the cluster (14) and we put $E_{IMF} = E_{IMF\ cluster}$ into (13), we get $m_{bound}^2 = m^2$.

[4]Due to the already mentioned mistake in previous publications [1, 2, 3] by a factor 2 in the binding energy, the incorrect value $g_t|_{phase\ transition} \simeq 1.24$ was previously quoted.

At this level we have included t- and u-channel Higgs and gluon exchange, both taken as massless particles, and we have only used as constituents the top and antitop quarks. We have not included the W or eaten Higgs exchange that could lead, as we shall see below, to partly b quarks among the constituents. Also we worked in the nonrelativistic or infinite momentum frame approximation, and so far we did not specify at what scale to take the g_t although we have put in essentially the perturbative QCD scale at m_t for the gluon coupling.

In the article of Kuchiev, Flambaum and Shuryak [5], these authors crudely take into account gluon exchange and u-channel exchange by taking the potential between two top quarks to be twice as big as the t-channel Higgs potential (1). Using this we can extract from Eq. (5) in their article the critical value of g_t needed to make the binding energy $< -H >$ equal to just half of the mass energy Nm_t (where $N = 12$ is the number of constituents)—as is required according to our infinite momentum frame formula (20) below. This critical value becomes $g_t|_{KFS} = 1.91$. This value should be compared to the just given value of 1.68 in (19) after it has been corrected for the many body effect performed in Appendix I, which gives $1.68 * (2.16)^{1/4} = 2.04$ to be compared with $g_t|_{KFS} = 1.91$. The small 6% difference is mainly due to the crude treatment of the gluon and u-channel correction. Rather than an increase in the t-channel Higgs potential by a factor of 2, we find that with the value (19) above for g_t the factor would be 1.70. With such a correction factor multiplying the t-channel Higgs potential, we get $g_t|_{KFS} = 1.91/\sqrt{1.70/2} = 2.07$ in close agreement with our many body corrected value of 2.04.

2.4 Justification of formal nonrelativistic mass-energy cancellation of half constituent mass to get zero mass bound state

We already argued that analyticity of the energy in the infinite momentum frame, or equivalently the bound state mass squared, suggested that we could formally use the nonrelativistic binding energy calculation and extrapolate it until the mass squared of the bound state becomes zero or even less than zero, if we wanted to obtain the phase transition value $g_t|_{phase\ transition}$ of the Yukawa coupling.

It is easy to see that the formal nonrelativistic requirement for this extrapolation in mass squared to make m_{bound}^2 zero means that the binding energy is adjusted to obey

$$\sum_i m_i/2 - E_{binding} = 0, \qquad (20)$$

rather than the intuitively expected requirement, which does not have the factor $1/2$ on the mass term. This factor $1/2$ came in from the Taylor expansion (15) of m_{bound}^2 in terms

of the binding energy $E_{binding}$.

We would now like to justify such a formal rule for calculating the critical g_t-value $g_t|_{phase\ transition}$. For that purpose, in Appendix B, we imagine writing the infinite momentum frame energy $E_{IMF\ cluster}$ for a cluster of "constituents" first in the case that the nonrelativistic approximation is valid. In this case we obtain the energy expression[5]

$$E_{IMF\ cluster} = p_z + \frac{1}{2p_z} \sum_i \frac{m_i}{x_i}(m_i + 2H_i). \tag{21}$$

Here H_i is the contribution of the i'th particle to the total nonrelativistic Hamiltonian $H = \sum_i H_i$ in the rest frame of the bound state:

$$H_i = \frac{\vec{p_i}^2}{2m_t} + \frac{1}{2} \sum_{j,\ i\neq j} V_{ij} \approx \frac{\vec{p_i}^2}{2m_t} + V_{total}, \tag{22}$$

where we approximate the interaction by a central potential V_{total}, while only half of the other 11 particles are present, and also include the u-channel and gluon exchange contributions by using Eq. (8). This approximation corresponds to taking the effective two particle interaction to be

$$V_{ij} \approx -\frac{A}{4\pi r_{ij}} \quad \text{where} \quad A = \frac{2(e_{t\bar{t}}^2 + 4g_t^2)}{11}. \tag{23}$$

Here r_{ij} denotes the distance between the ith particle and the jth particle.

We could now imagine that we want to use the expression (21), in order to obtain the critical value for the Yukawa coupling from the requirement of the bound state being of zero mass. This would mean that the term proportional to $\frac{1}{2p_z}$ should be zero to determine $g_t|_{phase\ transition}$. A symmetry argument between the different constituents—at least in the case of interest here in which all the constituents are the same type of particle—would suggest that we have to obtain this zero by all the operator factors $m_i + 2H_i$ being actually zero, in the sense of the nonrelativistic single particle Hamiltonians H_i having eigenvalues $-m_i/2$. If we believe this basic analyticity assumption argument, we have arrived at a justification for our rule of calculating the critical g_t coupling, according to which one shall require there be an eigenvalue for the binding energy equal to *half the mass m_i value*. In other words one shall find an "eigenstate" Ψ for which the equations

$$(\frac{m_i}{2} + H_i)\Psi = 0 \ \text{ for } i = 1, 2, ..., n \tag{24}$$

are satisfied. Let us here stress that, in this equation (24), the central potential approximation which we use in the single particle Hamiltonian H_i corresponds to the half-filled situation, so that $Z = \frac{11}{2}$ and the potential $\frac{1}{2} \sum_{j\ j\neq i} V_{ij}$ is replaced by V_{total}.

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[5]Note that this expression (21) agrees with Eq. (119) of Appendix B, when all the constituents move with the same speed in the z-direction so that $x_i = \frac{m_i}{\sum_j m_j}$.

If one wanted the physical Higgs field, one should rather ask for the Higgs potential felt by a constituent after all the other 11 constituents have already been put into the system. This would, in our concentration in the center approximation, give twice as strong a Higgs field as if one naively used the Higgs potential in H_i. But this is only true for the large r region, where one truly can expect all the particles generating the Higgs field to be at smaller r than the considered one. So for large r indeed one should multiply the deviation of the Higgs field from the usual VEV, as given by the Higgs potential in H_i, by a factor of 2. However, for an average distance $< r >$, about half the field producing constituents are farther away from the center and their contribution can crudely be ignored. So for $r \approx < r >$ this factor of 2 is compensated by the factor of $1/2$ corresponding to only getting the field from the constituents closer to the center. So here you get the Higgs field corresponding to the Higgs potential as present in the expression H_i. Further inside, corresponding to r less than $< r >$, the true field deviation from the usual VEV is even smaller than that corresponding to the Higgs potential in H_i.

The outcome of this discussion is that:

1) We argue that it is reasonable to use the nonrelativistic approximation as a rule that should lead to our wanted calculation.

2) It is important that in this rule one should get the zero mass bound state by requiring only half the mass be compensated by the binding. The other half should then in reality be canceled by the suggested analytic extrapolation.

3   Introduction of left-handed bottom quarks

We have so far left out the exchange of the weak gauge bosons but, with the estimate of the radius of the bound state given in Appendix C being of the order of $r_0 \simeq (\sqrt{4/3}m_t)^{-1}$, we should not necessarily ignore weak gauge boson exchanges. Actually in this section we shall only include these weak gauge boson exchanges in the approximation of letting the gauge couplings go to zero. At first you might think that in this limit we could totally ignore the exchange of the W and Z^0, but that is not true, because for the longitudinal components of these bosons there is then a zero in the inverse propagator due to gauge symmetry. In fact these longitudinal W and Z^0 components really represent the "eaten" Higgs components (see Appendix D). So what we shall really do is to replace the exchange of Z^0 and W by the exchange of their longitudinal components and postpone the discussion of the effect of their Coulomb fields until Secs. 10 and 11. Equivalently we can think of it as the exchange of the components of the Higgs other than the physical Higgs particle, which we already considered at length in the foregoing section.

In as far as the Higgs field has two complex components, of which we have in the

foregoing section only included the uneaten real part of one of them, there are three more real fields in the Higgs doublet, and these can be exchanged between the constituents in our bound states. The components eaten by the W are the charged fields and they will when exchanged from a top quark convert it to a b quark or oppositely.

For the understanding of the correction due to these exchanges, let us first note that we should have in mind that the particles which couple sufficiently strongly to be included in our approximation are as follows:

1) The *left-handed b* quark components.

2) The whole Higgs doublet (but we do not need the W and Z^0 fields proper, nor the photon field). Only the eaten components and the physical Higgs are included.

In this way we cannot properly have b quarks or anti-b quarks in our bound states, since they would all the time have to be represented by the right-handed top components whenever they need to be represented by right-handed components.

We shall further make the approximation that, whenever a pair of (left) $b\bar{b}$ quarks has been made by eaten charged Higgs exchange, then it soon gets again annihilated back to the usual situation of there being only top quarks and antitop quarks in our bound state. This assumption means that we only take into account that a right-handed top and an anti-right-handed top—which really must have left helicity—exchange an eaten Higgs between them and become a $b\bar{b}$ pair of bottom quarks, which then in the next interaction return back to become again a pair of right-handed top quarks.

For definiteness one could think of the self-energy diagrams for a combination of fields with the appropriate conserved quantum numbers to have an overlap with the bound state. Then, due to the summation over an infinite number of diagrams, a pole should be generated at $p^2 = m^2_{bound}$ corresponding to the mass squared of the bound state. In this section we shall use a box-diagram approximation, according to which the dominant self-energy diagrams are the ones in which the doublet propagators are restricted to circulate around box subdiagrams, with singlet right-handed top-quark propagators attached to the four vertices. The singlet right-handed top-quark propagators are not restricted and could, for instance, cross over each other forming a nonplanar diagram.

In the previous section we only included the "physical" Higgs component and only the left-handed top quark as particles that could come into these box diagrams. So we could think of the previous calculation as having used, for the box-diagram description of the scattering of right-handed top quarks, only those box diagrams in which the left-handed top quark and the physical Higgs components were included. Since the physical Higgs component is only one purely real part of one of the complex components of a Higgs doublet (which has two complex components equivalent to four real ones—meaning two purely real and two purely imaginary components), we must also imagine that in a

corresponding sense in this approximation of the previous section only the real part of the left-handed top-quark field was used.

By the inclusion of the left-handed bottom quark components and the three eaten real components of the Higgs, we have 4 times as many real components to exchange and to be represented as propagators in the box diagram we described. Actually all four combinations of Higgs and of left-handed bottom or top-quark real components that can circle around in the box diagram will give the same diagram contributions. Thus the effect of including the three eaten components and the left-handed quark components connected with them in the box diagram should simply be to increase the size of the box-diagram contribution by a factor 4, as may be checked in Appendix E. Now this box diagram is proportional to g_t^4, because each of its four corners contributes a top-quark Yukawa coupling. In Sec. 2.3 we estimated the top-quark Yukawa coupling needed to make the $6t + 6\bar{t}$ bound state to have zero mass to be $g_t = 1.68$. Naively, in our present approximation, this estimation could in principle have been made by using only the box diagram, but really to claim that would need some argumentation which we postpone to Sec. 4. This means that, in this approximation, the contribution to the binding energy from the box diagram with physical Higgs exchange would have to have the same value as in the previous section, in order to make the bound state massless. However we have seen that this box-diagram contribution should be increased by a factor of 4. This must mean that we should correct our predicted value of g_t^4 down by a factor 4, in order to obtain again, in the more correct calculation including the eaten Higgses, the massless bound state of $6t + 6\bar{t}$.

Thus we have now reached the estimate that the critical coupling g_t arranged to make the proposed bound state of $6t + 6\bar{t}$ to have zero mass becomes

$$g_t|_{phase\ transition} = 1.68/4^{1/4} = 1.19. \tag{25}$$

This estimate of the Yukawa coupling, giving the exact masslessness of the bound state of $6t+6\bar{t}$, was made using only the t- and u-channel gluon and Higgs exchanges. However, we made an oversimplified approximation with respect to the exchange of the eaten Higgses (really longitudinal W and Z^0 exchange), meaning that we considered deviations from there being only physical Higgs exchange inside certain box diagrams. A major point is that we have included the presence of left-handed b quark components as constituents rather than only top quarks. So we should perhaps not say that our state is exactly composed from $6t + 6\bar{t}$, since actually it is now considered possible to virtually replace a top and anti-top-quark pair by a bottom and antibottom quark pair.

It is now the idea to make a series of smaller corrections below to the approximations used to reach Eq. (25).

First, in Sec. 4, we shall discuss the correction to our box-diagram approximation coming from other closed loops of weak isospin doublet lines. However, before doing so, we consider the possible effect of diagrams involving interactions with the VEV of the Higgs field, represented by a Higgs-propagator symbol with a cross at one end (a tadpole). Because weak isospin is formally upheld in the Feynman rules, it follows that the couplings with the Higgs VEV have to come in pairs. We here want to argue that, when we precisely require the bound state to be exactly massless, these diagrams involving tadpoles must add up to zero.

An argument for this runs as follows: Clearly the sum over those diagrams having just two vacuum couplings will be proportional to the square of the Higgs VEV. Provided we ignore the direct dependence of the mass of the bound state on the Higgs mass (which according to Sec. 5 contributes a 5.2% correction to $g_t|_{phase\ transition}$), we expect, for dimensional reasons, that the bound state mass for fixed values of the coupling constants must be proportional to the Higgs VEV, except perhaps for very small renormalization group effects. But now, as we insist on looking for the zero mass case, there will be no dependence on the Higgs VEV. In turn that means that the total contribution to the change in the mass of the bound state, arising from the diagrams with two tadpoles, must be just zero.

Accepting this argumentation then the contributions arising from the insertion of one pair of tadpoles into the diagrams should at the end add up to just zero. Really you can argue similarly that the diagrams with four tadpoles and so on would also cancel out.

Finally we have argued that, for our specific project of finding that g_t value for which we can have a massless bound state, we can ignore the tadpole diagrams and thus concentrate on those diagrams in which all the isospin doublet propagators form loops of longer or shorter lengths. We assumed above that it is the very shortest loops which matter most, but in the next section we shall discuss corrections to this box-diagram approximation.

4   Corrections to the eaten Higgs exchange force and thereby bottom quark admixture

Actually it is not correct that the left-handed bottom or top quarks would only circle around in box diagrams. In order to obtain an idea as to how much this box-diagram approximation has to be corrected, let us imagine a diagram being written down for how the bunch of 12 top or antitop quarks propagate with mutual interaction under what really corresponds to the development of the bound state. As in Sec. 2, there are interactions between any of the top or antitop constituents and any other one among them. We

imagine constructing the diagram by drawing a series of 12 top-quark lines representing chains of top-quark propagators. Next we divide these lines up into propagators while decorating them with exchanged particle propagators going from one of the lines in the chains to one of the other ones. At first we imagined that we had top-quark propagators representing both right-handed and left-handed components. But it is actually rather easy to imagine that, in our Feynman rules, we make different propagators for right-handed and left-handed components so as to introduce one propagator for left and another one for right. Then the Higgs vertices all the time connect a left to a right, while the gluon vertices oppositely couple left to left and right to right.

After having imagined the notation with left-handed and right-handed t propagators being treated as different particles, we can rather easily introduce the left-handed b quarks by imagining that we allow the left propagator to be treated as if it had both a t and a b component built into it. So the left propagators represent simultaneously two types of particles, b and t, while the right propagators are kept unchanged and only represent the right-handed t quark. At the same time we have to introduce also the eaten components for the Higgs propagator, but that we do analogously by just deciding that now the Higgs-propagator symbol stands for both complex components being propagated. In other words we just reinterpret the diagram to include the eaten components and the left-handed b quark also. The diagram will look formally the same as when we just separated the diagram into left and right t propagators without any b quarks.

In this latter notation we can follow the propagation of the doublets through the diagrams. That is to say we can follow chains of propagators, which are either left t and b combined propagators or the full Higgs propagators. Since these two types of propagators are doublets under weak isospin, while the right t propagators and the gluons are weak isospin singlets, it is clear that the chains of doublet propagators cannot end anywhere in the interior of the diagram. Ignoring the case of external lines being doublets, they would have to form loops inside the diagram considered. In Sec. 3 we actually made the approximation that these loops of doublets would always be box loops having only four propagators along the loop. But that is of course by no means guaranteed.

There is however an argument that the small box loops of doublets might be favored compared to more extended doublet loops: We get our factor 4 increase in the value of the whole diagram, due to the inclusion of the eaten components and the b quark, for each doublet loop that can be found in such a whole diagram. For a given number of left and Higgs propagators one thus gets the biggest increase factor—i.e., more factors of 4—by putting the doublet propagators into as many loops as possible. That will then mean to put them into loops with as few propagators as possible around them. But such loops correspond to box loops. Since the propagators around a doublet loop must, namely,

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alternate a left-handed quark with a Higgs back and forth, we must always have an even number of propagators in such a loop of doublets. So four is the minimum nontrivial number of propagators in the loop.

Were it not for such an effect of a somewhat higher factor for the small doublet loops, the doublet loops could be rather long because they would be obtained by combinatorically taking random diagrams.

We now want to correct for the fact that our assumption of there being only the box loops of doublets overestimates the effect on the correction to $g_t|_{phase\ transition}$ from the inclusion of b quarks. In fact we used above that the factor 4 per doublet loop could be compensated for by a corresponding reduction in g_t^4 by the factor 4. But now, since many of the doublet loops can be longer than 4 propagators around but rather on the average n propagators around, the correction should instead have been that g_t^n be reduced by a factor 4. That of course would lead to the change

$$g_t|_{phase\ transition} = 1.68/4^{1/n}, \tag{26}$$

where we now have to estimate an appropriate average for the quantity n.

A doublet loop with n vertices along it has n doublet propagators. So, compared to the box doublet loops, it has per loop $(n-4)/4$ too few factors of 4 due to the summing over the different components that can propagate around the loop. This gives for such doublet loops a suppression weight factor $4^{-(n-4)/4}$. This means that if you compare the contribution for one diagram and one modified locally in the diagram reorganizing it so as to replace $n/4$ box loops by one n-"propagator" loop[6] of isodoublets in the local region considered, then the magnitude of the square of this modified diagram will be $(4^{-(n-4)/4})^2$ times the corresponding square of the replaced diagram. Let us suppose that statistically, ignoring the extra factors for the isodoublet loops, the distribution in a random (typical) diagram of the loop size n is smooth. This distribution of the number of propagators around the isodoublet loops is briefly discussed in Appendix F. Taking this distribution to be flat and essentially constant for the first few n values, we obtain that the probability distribution of n (on random diagrams weighted with their magnitude squared) would go as $(4^{-(n-4)/4})^2$. If you somehow weighted with amplitude rather than the squared amplitude, the "distribution" would only go as $4^{-(n-4)/4}$.

We may see that this means, in the example of, e.g., the six sided doublet loop, that its weight factor is $4^{-(6-4)/4} = 1/2$ in amplitude. But in probability the six sided loops are suppressed rather by $(4^{-(6-4)/4})^2 = (1/2)^2 = 1/4$. Thinking of the Feynman diagrams

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[6]Here we count a series of doublet propagators, which reduce to a single propagator when gluon propagators are ignored, as a single "propagator". Really the easiest way of thinking about this is to say that we totally ignore the gluons in this calculation of the backcorrection to our box-diagram approximation of Sec. 3.

as adding up with *random phases*, the resulting sum of a lot of Feynman diagrams would statistically get a magnitude corresponding to adding them in quadrature rather than simply adding real positive numbers with the size of the series of diagrams. We indeed take it that the weighting of the importance of loops of a given number of doublet propagators n shall be counted as proportional to the squared quantity $(4^{-(n-4)/4})^2$ rather than to $4^{-(n-4)/4}$ itself. It is then easily seen that the relative importance of the contributions of the loops with the series of n values (being $n = 4, 6, 8, 10, ...$) form the series of terms

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{4}{3}. \tag{27}$$

It follows that, instead of all the correction factors to $g_t|_{phase\ transition}$ in Sec. 3 being $4^{-1/n} = 4^{-1/4}$ (coming by thinking of just box loops having $n = 4$), we get the following series of correction factors corresponding to the series of terms in (27)

$$4^{-1/4} \ , \ 4^{-1/6} \ . \ 4^{-1/8} \ , \ 4^{-1/10} \ , \ \cdots . \tag{28}$$

Compared to the correction as made in Sec. 3 to $g_t|_{phase\ transition}$ (multiplicatively), we get instead a further correction—which is really correcting back for the fact that we have overcorrected—by

$$\frac{1 * 1 + \frac{1}{4} * 4^{-1/6+1/4} + \frac{1}{16} * 4^{-1/8+1/4} + \cdots}{1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots} \approx 1.04. \tag{29}$$

We have thus crudely estimated that the *b*-inclusion correction of Sec. 3 has to be modified, so as to put the estimate of the critical $g_t|_{phase\ transition}$ *up* by 4%.

For a very big number N of Higgs components to exchange and a corresponding number of left-handed quark components, the smallest closed loop for the circulation of the weak isospin will be favored. This is because we get a factor of N for each closed loop. Then for a self-energy diagram with a given number of doublet propagators, we get the largest number of factors of N by using the diagram with the largest number of loops. For large N it follows that the box-diagram approximation will dominate. This number N is 4 in the true standard model as already discussed above. For N being small, however, we have no guarantee for this box-diagram dominance at all and indeed this is the situation for which our calculations in Sec. 2 were performed, namely, for $N = 1$. So we do not yet really have a good argument for how this Sec. 2 calculation can be related to the higher N cases.

Let us now consider the dependence of $g_t|_{phase\ transition}$ on N, by introducing a parameter $n(N)$ giving the typical doublet loop size in our complicated Feynman diagrams, so that

$$g_t|_{phase\ transition} = g_0/N^{1/n}. \tag{30}$$

Here g_0 does not depend on N. For large N the box diagram dominates and we clearly have $n(N) \to 4$. It is also clear that the denominator $N^{1/n} \to 1$ for $N = 1$. But then we can essentially *interpolate* the denominator to be approximately $N^{1/4}$ for all N. Therefore we can effectively calculate *as if* Eq. (30) were replaced by

$$g_t|_{phase\ transition} \approx g_0/N^{1/4}. \tag{31}$$

and thereby justify Eq. (25) as a good approximation, because it is a good interpolation of the general formula.

We estimate that the uncertainty in this interpolation formula is of order $\pm 7\%$. Combining this $\pm 7\%$ error with an estimated error of $\pm 4\%$ on the correction calculated above, we obtain a total error of $\pm 8\%$ Thus our final result for the correction to $g_t|_{phase\ transition}$ is to increase it by $4 \pm 8\%$.

5 Correction due to Higgs mass

In Appendix C we estimate the Bohr radius of our bound state of $6t + 6\bar{t}$ in the critical coupling case to be $r_0 \approx (\sqrt{4/3}m_t)^{-1}$ and thus we see that with a Higgs mass of 115 GeV, which we use in this paper, the effect of the Higgs particle having a nonzero mass would not be so dramatic for our calculations. It is however not just this argument of the radius being small which is the true reason for our correction, due to the nonzero mass of the Higgs, being only a small correction. Rather the argumentation is the following:

As we go into the interior of the bound state we find the Higgs field due to all the top and antitop quarks around. These fields have such a sign as to mean that, in reality, the normal vacuum value of the Higgs field is diminished in the interior of the bound state. So we shall not use the Higgs mass for the normal vacuum, but rather some effective Higgs mass on the background of the Higgs field inside the bound state. This effective Higgs mass squared is extracted from the second derivative of the effective potential for the Higgs field at the value at which we want to "work." To a good approximation the Higgs field effective potential is given as a fourth order polynomial

$$V_{eff}(\phi_h) = -\frac{1}{2}|m_{hbare}|^2|\phi_h|^2 + \frac{\lambda}{8}|\phi_h|^4. \tag{32}$$

So the physical Higgs mass squared is related to the second derivative of this expression at the minimum, where the value of the field ϕ_h must be fitted to 246 Gev.

But now it is obvious that the second derivative and thus the effective Higgs mass squared becomes smaller for lower values of the field ϕ_h, where we want to extract this second derivative. Since in the interior of the bound state the Higgs field is supposed to be smaller, then the Higgs mass to be used there actually also becomes smaller than in the

normal vacuum outside the bound state. In this section we shall at first ignore gluonic contributions to the binding energy. Then we estimate that the Higgs field inside the bound state deviates so strongly from the one in the normal vacuum that even the sign of the full Higgs field—the vacuum value plus the field contributed by the quarks—tends to be inverted in the most interior part of the bound state. Thus, in most of the interior of the bound state volume, the second derivative is much smaller than in the normal vacuum or even negative. The latter corresponds formally to an imaginary effective Higgs mass. So in an averaged way the Higgs mass squared is, to first approximation, an average of both negative and positive contributions.

Really we should split up the volume of the bound state into a region—the more interior region—with imaginary effective Higgs mass and a more exterior region in which the Higgs mass is real, but even there the numerical value of the mass is diminished.

In order to get an idea about how strong the Higgs field should be in the interior of the bound state, we may use the virial theorem. According to the virial theorem, in the nonrelativistic approximation which we use with a $1/r$ potential, the magnitude of the potential energy has to be twice as big as the total binding energy. Now we have precisely decided to adjust the top-quark Yukawa coupling, so as to make the total binding energy per constituent numerically equal to half its mass $m_t/2$. This then means that the potential energy per top quark should be $-m_t$ in the potential $\frac{1}{2}\sum_{j,\ j\neq i} V_{ij}$ felt by a constituent only feeling half the other 11 quarks.

We can understand that the change in energy of a top quark, resulting from the reduction of the full Higgs field down from its normal vacuum value to zero, would remove the mass and thus correspond to a change by $-m_t$. So in the potential due to only half of the constituents of the bound state we need just this effect, meaning that the Higgs field should be zero (in the approximation of ignoring the gluons) at the typical distance from the center. Thus, taking into account that only half the constituents are inside the average radius, we estimate that the Higgs field at this average radius distance actually vanishes, $\phi_h|_{at\ average\ pos.} = 0$. In the very most interior of the bound state the effective Higgs mass is not so important, since the distances are anyway small compared to the Compton wave length of the Higgs. On the way out from the center, the effective Higgs mass is small or even imaginary and only in the outskirts of the bound state does it take on approximately its normal value.

Thus, at the average radius $<r> = 3/2 * r_0$, the potential energy per top quark should be equal to $-m_t$, when we compensate the total mass of the bound state and make it zero by letting the binding energy be $m_t/2$ per constituent. This means that the Higgs field is zero, $\phi_h|_{at\ average\ pos.} = 0$, at this average radius $<r> = 3/2 * r_0$.

Now the effective potential for the Higgs field has an inflection point—i.e., second

derivative zero—when its value is $1/\sqrt{3} = 0.58$ times the value in the normal vacuum $<\phi_h>_{normal} = v$. This inflection point value of the Higgs field thus deviates from the normal vacuum expectation value of the Higgs field by $(1 - 1/\sqrt{3})v = 0.423v$, while the average value of the Higgs field reached at $r = <r> = 3/2 * r_0$ deviates by v from the normal value. Since, in the first approximation, the potential felt by the quark in the bound state goes down inversely with the distance r from the center, i.e., as $\propto 1/r$, the inflection point is reached when $1/r$ has fallen by a factor of $1/0.423 = 2.37$ compared to $1/<r> = 2/(3r_0)$. This means that the inflection point is reached at the distance $r_{inflection} = 3/2 * 2.37r_0 = 3.55r_0$.

5.1 Correcting the Higgs field strength in the interior due to the force being partly gluonic

In Sec. 2.2 we calculated that approximately $1/3$ of the force responsible for the binding of the top and antitop quarks in our bound state was due to gluonic rather than Higgs exchange. Thus the binding energy from the Higgs exchange by itself should only make up approximately $(2/3)^2 = 4/9$ of the total binding energy. Rather than having the potential energy per top quark equal to $-m_t$ due to the Higgs field being zero at the average distance from the center, as estimated above, we should instead have that this average value of the Higgs field should be $\phi_h|_{at\ average\ pos.} = (1 - 4/9)v = 5v/9$. As we shall see in Appendix G, we estimate that the field strength measured as the deviation from the normal vacuum expectation value v, i.e., $-(\phi_h - v)$, reaches a value at the very center of the bound state which is about $3/2$ times the average deviation. Hence, when the average of the Higgs field deviation is $4v/9$, this maximal deviation—or the maximal field strength due to the top and antitop quarks—will be $3/2 * 4v/9 = 2v/3$. Thus the actual value of the Higgs field in the center of the bound state is $(1 - 2/3)v = v/3$, meaning that it is one-third as strong as in the usual vacuum. Hence the effective Higgs mass remains imaginary all the way into the center, after we have passed deep enough into the bound state for the value of the Higgs field to fall below its value $v/\sqrt{3}$ at the inflection point.

The conclusion is that, closer to the center than the distance at which the field ϕ_h has the strength $v/\sqrt{3}$ corresponding to the inflection point in the Higgs effective potential, we have an imaginary effective Higgs mass. We will now consider the real and imaginary effective Higgs mass regions separately.

5.2 The real Higgs mass region

The only region in which we get a real effective Higgs mass is at distances so far from the center that the value of the Higgs field has risen above the inflection point value of $v/\sqrt{3}$.

So let us first consider the Higgs field in this region, where it is numerically bigger than at the inflection point.

According to the above discussion, the average value of the Higgs field in the region of the constituents of the bound state should be $5v/9$. We take it that this value is reached at the average position given by $r =< r >= r_0 * 3/2$, where r_0 is the Bohr radius. Since the Higgs effective potential has an inflection point when the Higgs field takes on the value $\phi_h = v/\sqrt{3}$, this must occur in the bound state when the distance r from the center has been increased relative to $< r >= 3/2 * r_0$ by a factor $\frac{1-5/9}{1-1/\sqrt{3}}$. Thus, as one moves out from the center of the bound state, the inflection point in the Higgs effective potential is passed at the distance $r = r_{inflection} = \frac{1-5/9}{1-1/\sqrt{3}}* < r > = 1.052* < r >= 1.58 r_0$. With the probability distribution in r taken to be $\propto \exp(-2r/r_0)r^2 dr$, the probability for a quark being outside the distance characteristic of the inflection point $r_{inflection}$ becomes crudely

$$\frac{\int_{r_{inflection}}^{\infty} \exp(-\frac{2r}{r_0})r^2 dr}{\int_0^{\infty} \exp(-\frac{2r}{r_0})r^2 dr} \tag{33}$$

$$\approx \exp\left(-\frac{2r_{inflection}}{r_0}\right)\frac{\int_{r_{inflection}}^{\infty} \exp(\frac{-2(r-r_{inflection})}{r_0})r_{inflection}^2 dr}{\int_0^{\infty} \exp\left(-\frac{2r}{r_0}\right) r^2 dr} \tag{34}$$

$$= \exp(-2r_{inflection}/r_0)\frac{\Gamma(1)r_{inflection}^2}{\Gamma(3)(r_0/2)^2} \tag{35}$$

$$= \exp(-3.15)*3.15^2/2 \tag{36}$$

$$= 21.2\%. \tag{37}$$

Since the distance $r_{inflection}$ from the center to the inflection point field value is only 5% greater than the average distance $< r >$, the Higgs exchange Coulomb potential at $r = r_{inflection}$ is only reduced by a factor $< r > /r_{inflection} = 0.951$ compared to its value at the average distance. So the effect of even an order of unity change of the potential for $r > r_{inflection}$ (due to the Higgs mass effect) could at most change the average of the overall binding potential by the order of $0.951 * 21.2\% = 20.2\%$. At $r = r_{inflection}$ the probability distribution $\propto \exp(-2r/r_0)r^2 dr$ of the quarks has a logarithmic derivative of $(2/r_{inflection} - 2/r_0) = (2*0.951/ < r > -3/ < r >) = -1.098/ < r >$. So the range, over which we have a significant part of the probability, goes outside $r = r_{inflection}$ only by about a distance of the order of $< r > /1.098$. In that range the effective Higgs mass squared grows away from its starting value of zero at $r_{inflection}$. By using a linear Taylor expansion in ϕ_h, we estimate that the effective Higgs mass squared reaches $\frac{(1/1.098+1/0.9511)^{-1}}{1.098} = 0.464$ of its final value at infinite distance. The value of the infinite distance Higgs mass is the physical Higgs mass m_h, which we take to be 115 ± 50 GeV in this article. So the effective Higgs mass in the region of interest is $m_{h\,eff} = (115 \text{ GeV}) * \sqrt{0.464} = 78.3$ GeV.

The range over which this Higgs mass is active is about $< r > /1.098$, so that the correction factor, converting the Coulomb potential into a Yukawa potential, becomes $\exp(-\frac{m_{h\ eff}*3r_0}{2*1.098}) = \exp(-3\frac{m_{h\ eff}/(\sqrt{4/3}m_t)}{1.098*2}) = \exp(-3\frac{78.3/(172.6\sqrt{4/3})}{2.196}) = \exp(-0.54)$. However this 54% correction only applies to quarks at distances $r > r_{inflection}$ from the center of the bound state. So the percentwise correction to the total potential, due to the Higgs mass in the real Higgs mass region, is 20.2% of 54% = 10.9%. This effect gets doubled when calculating the binding energy, because the radius varies with the strength of the potential. However, since it is only for the Higgs part of the potential, it should also be reduced by a factor 2/3. Finally then we are interested in this paper in calculating the coupling $g_t|_{phase\ transition}$, which is extracted from the fourth root of the binding energy. So, at the end, this correction leads to an *increase* in the phase transition coupling, needed to get just zero mass for the bound state, by $2 * \frac{2}{3} * \frac{10.9}{4}\% = 3.6\%$.

5.3 The imaginary Higgs mass region.

As we have just seen the effective Higgs mass is imaginary in the region of greatest relevance for the binding of the top quarks and antitop quarks, namely, from $r = 0$ out to where the Higgs field takes on the inflection point value at the distance $r_{inflection} = 1.58r_0$ from the center. In Appendix G, we crudely estimate the average effective Higgs mass squared, in this region $0 < r < r_{inflection}$, to be $m^2_{h\ eff} = -m^2_h/12$.

The most important place to get effects from this effective imaginary Higgs mass is from the very most central region out to the average distance of the quarks and antiquarks feeling the potential, which must crudely be at the distance $r = < r > = \frac{3r_0}{2}$. The usual Yukawa potential having the form $\propto \exp(-m_h r)/r$ should formally be replaced by a form $\propto \exp(-i|m_{h\ eff}|r)/r$ in the imaginary effective Higgs mass region. However it should be real in as far as the Higgs field is "real" and, since the sign of the i in the exponent is ambiguous, we actually have to take

$$\phi_h \propto \cos(|m_{h\ eff}|r)/r \tag{38}$$

in the effective imaginary Higgs mass region.

Actually it is not difficult to see that an expression of this form obeys the Klein-Gordon equation with a tachyonic mass—i.e., $m^2_{h\ eff} < 0$. Requiring the Higgs field to be given by the Coulomb, i.e., massless, potential in the immediate neighborhood of the particle emitting it, we also see that the only solution to the Klein-Gordon equation with this boundary condition becomes the cosine form just presented.

We take the averaged effect of the Higgs field on the binding to be approximated by the effect at the average distance $< r >$. This means that the correction, due to the effective tachyonic Higgs mass $m_{h\ eff}$ being imaginary, will become a factor $\cos(|m_{h\ eff}| < r >)$

in the attractive Higgs exchange potential between two (anti-)quarks. Since the latter is proportional to g_t^2, this means that we effectively replace g_t^2 by $g_t^2 \cos(|m_{h\ eff}| < r >)$. That will in turn mean that the g_t value needed to achieve a certain condition for the binding—in our case that we bind just so strongly as to make the $6t + 6\bar{t}$ bound state massless—will have to be increased by the factor $\sqrt{\cos(|m_{h\ eff}| < r >)}^{-1}$. In other words

$$g_t|_{phase\ transition} \rightarrow g_t|_{phase\ transition}/\sqrt{\cos(|m_{h\ eff}| < r >)} \tag{39}$$

$$= g_t|_{phase\ transition}/\sqrt{\cos\left(\frac{m_h}{\sqrt{12}} * \frac{3}{2}r_0\right)} \tag{40}$$

$$= g_t|_{phase\ transition}/\sqrt{\cos\left(\frac{3m_h}{8m_t}\right)} \tag{41}$$

$$= g_t|_{phase\ transition}/\sqrt{\cos(0.250)} \tag{42}$$

$$= g_t|_{phase\ transition} * 1.016. \tag{43}$$

showing that $g_t|_{phase\ transition}$ is increased by 1.6%. Here we assumed the physical Higgs mass to be $m_h = 115$ GeV and we used the crude estimate $|m_{h\ eff}| = m_h/\sqrt{12}$ from Appendix G. For the Bohr radius of the bound state, we took $r_0 \approx (\sqrt{4/3}m_t)^{-1}$ from Appendix C. The average radius is, of course, $< r > = 3/2 * r_0$. Also we used the experimental value [4] of $m_t = 172.6$ GeV for the top-quark mass.

Combining this with the correction from the positive effective Higgs mass region of 3.6%, we get the total correction from the Higgs mass not being zero to be a $1.6\% + 3.6\% = 5.2\%$ increase in the value of g_t needed for the phase transition. We estimate a theoretical uncertainty of $\pm 2\%$ in this result. In order to take into account the ± 50 GeV error in the Higgs mass, we have repeated the above calculation for a Higgs mass of 165 GeV. We find the total correction in this case to be an increase of 8.5% in the critical value of g_t. So we conclude that, for a Higgs mass of $m_h = 115 \pm 50$ GeV, we obtain a total correction of $5.2\% \pm 3.3\%$. Combining in quadrature this 3.3% uncertainty arising from the error on the Higgs mass with the estimated theoretical uncertainty on the calculation of 2%, we finally obtain the value $5.2\% \pm 4\%$ for the increase in the value of g_t needed for the phase transition.

6   s-channel exchanges

We only calculated the contributions to the binding energy from the t-channel and u-channel exchanges above because:

a) These contributions are somewhat easier to calculate in the Bohr atom approximation.

b) We believe that the s-channel contribution will be relatively smaller due to the effect that, in an s-channel exchange, a quark and an antiquark together with their associated binding energy are virtually missing from the bound state. This leads to an extra suppression of the binding energy from the s-channel exchange.

In the present section we shall estimate the extra binding, due to the s-channel exchange of both Higgses and gluons.

6.1   Crude channel symmetry estimation of s-channel contribution.

First we shall make an estimate of the binding energy caused by the s-channel effect—let us first consider just the Higgs exchange—by thinking of an effective four quark interaction term. We then compare the s-channel contribution to the t-channel and u-channel contributions in such a formalism.

The plan is first to imagine a situation in which we could ignore the masses of the quark and antiquark, interacting via the virtual annihilation and recreation mechanism described by s-channel scattering. The energy can then be chosen so that there would be a symmetry between all three channels (s, t and u), apart from the selection rules. In this situation the dominant 4-momenta for the quark (anti-)quark scattering comes from the 3-momenta arising from the Heisenberg uncertainty in the momentum, which follows from the geometrical extension of the wave function for the quarks and antiquarks.

We may think of evaluating the binding energy, by taking the expectation value of an operator corresponding to the Feynman diagram for the Higgs exchange between a quark and an antiquark in one of the three channels (s, t or u). Such an expectation value of a lowest order scattering operator should then be the change in energy due to this interaction. Here we do not take into account that, after the inclusion of some interaction, one should also adjust the ground state wave function (e.g., the radius of the bound state). We now imagine an artificial arrangement of "small" energies, replacing the ones due to the quark masses, such that on the average the 4-momenta through the three channels (s, t and u) are arranged to be the same[7]. Thereby the propagators in these different channels will also be the same and thus the diagrams, when averaged, will give the same numerical values, as long as they are not simply forbidden by selection rules. This means that they would give equal contributions to the binding energy. It is these imagined momentum distribution configurations, which we want to use for estimating the size of the s-channel contribution to the binding energy relative to that from the t

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[7]This same value for the three quantities s, t, u is of course not at all consistent with the nonrelativistic situation, and strictly speaking it is even in the unphysical region in the Mandelstam diagram.

channel. Then we must correct for the fact that these artificially arranged 4-momenta get modified, when we instead take the external 4-momenta to contain the quark masses in the nonrelativistic situation. Furthermore, we must take into account the effects of the lack of binding energy to the other quarks, during the virtual time in which the pair of scattering quarks is absent from the bound state.

Let us denote by B the binding energy due to an allowed t-channel exchange between two quarks, which is achieved without changing the bound state wave function and is hence proportional to g_t^2 rather than to g_t^4. Then this binding energy B is indeed the expectation value of the operator connected with the t-channel exchange diagram for the scattering of the two quarks.

In the artificial situation proposed above, we arranged the energy components of the four-momentum distributions for the quarks, so that there was a symmetry between the three channels with respect to these four-momentum distributions. This then means that not only would the u-channel and t-channel interactions, counted in the same way, lead to the same binding B, but even the s channel would give the binding B in the artificial situation.

Next we must estimate the change in the binding B, when we include the correct rather than the artificial external 4-momenta. The idea is that this makes no difference as far as the three momentum is concerned. The major effect comes from the inclusion of the correct mass energies and from the lack of binding to the other quarks in the bound state during the s-channel quark scattering. Thus there is no difference—at least in the nonrelativistic approximation—to the 4-momenta in the u channel and the t channel. These u and t channels contribute a binding energy B by definition, and B is not changed relative to the artificial kinematical situation by including the nonrelativistic masses into the energies. So we only need to get the correct replacement for B for the s-channel diagrams.

We now need to estimate the correction to the s-channel propagator, by replacing the s-channel propagator with the artificial four-momentum going through it by the one having the correct four-momentum (mainly mass energy) going through it instead. Now the artificial four-momentum going through the s channel was precisely made up to be just the same as what goes through the t channel in the t-channel diagram. So really we ask for the ratio of the s-channel propagator in the true s-channel diagram to the t-channel propagator in the t-channel diagram. Then we can correct the binding energy, appropriately taken from the t channel, by this factor and thereby obtain the binding energy due to the corresponding s-channel exchange term.

In order to perform this correction, we need to estimate not only the binding B, which we have essentially already done in previous sections, but also the average of the square

of the four-momentum going through the t-channel propagator.

6.2 Estimate of size of average t-channel momentum in propagator

The average three momentum squared $\vec{q}^2$ in the t-channel propagator is achieved as the sum of the momentum distributions of two of the Higgs emitting quarks—really the same quark before and after the emission. Now we may easily estimate the expectation value of the $\vec{p}^2$ distribution for the quark in the bound state, using the virial theorem and the binding energy. In fact we have from Eq. (24) that

$$m_t/2 = \text{binding energy} = -(V_{potential} + < \vec{p}^2/(2m_t) >). \tag{44}$$

As discussed in Sec. 5, it follows from the virial theorem that

$$V_{potential} = -2 < \vec{p}^2/(2m_t) >, \tag{45}$$

which means that

$$< \vec{p}^2/(2m_t) > = m_t/2, \tag{46}$$

and hence

$$< \vec{p}^2 > = m_t^2. \tag{47}$$

So, since the t-channel exchange goes between a quark to quark transition vertex and another one, the probability distribution for the momentum squared in the propagator should really be the product of the distributions appearing from the two emissions. In the Gaussian approximation the product distribution will have the spread $< \vec{q}^2 >$ obtained by adding the inverse $< \vec{q}^2 >$'s, i.e., $< \vec{q}^2 >^{-1}$ for the two distributions multiplied. These emission distributions in turn have, in Gaussian approximation, the average of the $\vec{q}^2$ given as the sum of that for the quark before and that after the emission. It is easy to see that we then end up having the four or equivalently three momentum squared in the t-channel propagator being the same as the distribution of $\vec{p}$ for a single quark in the wave function. In other words we obtain the propagator momentum squared average

$$< \vec{q}^2 > = m_t^2. \tag{48}$$

6.3 Naive calculation with just the quark masses

If we just calculate naively, according to the prescription suggested, we should now simply insert the crude nonrelativistic approximation $2m_t$ for the s-channel propagator four-momentum value in the time direction, which dominates. This would mean a decrease

of the s-channel propagator by a factor 4 compared to the one in the artificial situation or equivalently relative to the t-channel one. This is only so simple because we ignore both the Higgs mass and the lack of binding energy coming from the quark-antiquark pair during their virtual annihilation time. This result means that the binding energy due to the s channel is reduced from B down to $B/4$.

In the following part of this section we shall correct this naive s-channel binding energy expression $B/4$, by taking into account the very strong interaction which the considered quarks, annihilating into the Higgs, have with the other quarks in the bound state. In our case, in which the binding cancels the mass energy, of course this interaction must be very significant. Although including such effects is in principle higher order and really corresponds to calculating loop diagrams, we indeed need to include them at least crudely. We shall perform these corrections in a couple of steps:

1) We shall consider the Higgs relativistic Feyman propagator from a nonrelativistic quantum mechanical second order perturbation theory point of view, interpreting it to have two physically different factors in the denominator; see Sec. 6.4.

2) We shall take into account and estimate the extra energy contribution accompanying the Higgs, due to the change in the binding of quarks inside the bound state; see Sec. 6.5.

6.4 Comparing nonrelativistic perturbation with Feynman propagator

It is well known that the relativistic Higgs-propagator is

$$\text{prop}(p) = \frac{i}{p^2 - m^2} = \frac{i}{(p^0 - E(\vec{p}))(p^0 + E(\vec{p}))}. \tag{49}$$

This is made in a normalization of the Higgs field ϕ_H given by the expression

$$\int \phi_H^\dagger \overset{\leftrightarrow}{\partial_0} \phi_H d^3 x = 1. \tag{50}$$

This normalization deviates from the simple nonrelativistic one:

$$\int \phi_{H\ nr}^\dagger \phi_{H\ nr} d^3 x = 1. \tag{51}$$

For approximate energy eigenstates with energy E_{Higgs}, this implies the following relationship between the relativistically and nonrelativistically normalized fields:

$$\phi_H = \frac{1}{\sqrt{2E_{Higgs}}} \phi_{H\ nr}. \tag{52}$$

The interaction energy density of the Yukawa term in the Lagrangian becomes

$$\mathcal{H}_{Yukawa} = -g_t(\bar{\psi}_{tR}\phi_H^\dagger \psi_{tbL} + H.c.) \tag{53}$$

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[see Eq. (99) in Appendix A for notation], where the field ϕ_H is the relativistically normalized field. Thus, in nonrelativistic notation, this Hamiltonian density would rather look like:

$$\mathcal{H}_{Yukawa} = -\frac{g_t}{\sqrt{2E_{Higgs}}}(\bar{\psi}_{tR}\phi_{H\ nr}^{\dagger}\psi_{tbL} + H.c.). \tag{54}$$

Now, according to usual nonrelativistic second order perturbation theory, one has the correction to say the energy of the ground state $|gs>$ from this second order effect:

$$<gs|\int \mathcal{H}_{Yukawa}d^3x|Higgs><Higgs|\int \mathcal{H}_{Yukawa}d^3x|gs>/(E_{Higgs}-E_{gs}). \tag{55}$$

For example, say we wanted to consider the change in energy of a quark-antiquark pair due to s-channel Higgs exchange, then E_{gs} would be the energy of the unperturbed pair and E_{Higgs} would be $E(\vec{p}) = \sqrt{m_h^2 + \vec{p}^2}$. In Eq. (49) the denominator factor $p^0 - E(\vec{p})$ is thus to be identified with the denominator in the nonrelativistic perturbation correction (55), i.e.,

$$p^0 - E(\vec{p}) = -(E_{Higgs} - E_{gs}). \tag{56}$$

In the nonrelativistic notation, using (54), and for p^0 close to the on-shell energy of the Higgs, the matrix elements in (55) each contain an extra denominator $\sqrt{2E_{Higgs}}$ compared to the relativistic notation, which we can transfer to the propagator. In this way we get a propagator to be used, together with the formal relativistic notation matrix element, without such a denominator,

$$\frac{-i}{E_{Higgs}-E_{gs}} * \frac{1}{\sqrt{2E_{Higgs}}\sqrt{2E_{Higgs}}} = \frac{i}{(p^0-E(\vec{p}))(2E_{Higgs})} \approx \frac{i}{(p^0-E(\vec{p}))(p^0+E(\vec{p}))}$$
$$= \text{prop}(p). \tag{57}$$

It will be important for us to use this physical interpretation of the two different factors in the denominator of the relativistic propagator, when in the next section we shall take into account the very strong interaction of the quarks, which annihilate into the Higgs, with the rest of the quarks in the bound state.

6.5 Extra energy in the intermediate state

The important effect of the strong interaction, between the two annihilating quarks and the other quarks, is that the energy of the remaining 10 quarks (really 5 quarks and 5 antiquarks) may be changed drastically by the absence of the annihilated quarks. This change in energy means that the energy of the intermediate state—which is talked about here as the Higgs state—is actually shifted relative to the Higgs energy proper to an effective Higgs energy including this interaction energy change.

When we want to include the missing binding energy of the annihilated pair together with them into the calculation, one should strictly speaking consider the whole process

described by an effective loop Feynman diagram, in which the bound state of 12 particles (the t ball) is split up into a Higgs and a "core" consisting of a bound state of 10 constituents. The loop vertex should then really be a description of the annihilation process coupling to the emitted Higgs. If we indeed went into the details of the estimation of such a loop, we would have to integrate over a loop energy, p^0. The integrand would have poles coming from both the Higgs and the core (i.e., the 10-constituent bound state). In fact we propose to look at the contributions from near these poles as two terms to be calculated separately. In order to avoid going into the details of loops, we shall however make another presentation, in which we instead only talk about tree diagrams. The price, however, is that now we must vary what state we take as the background (or one could say the vacuum), in evaluating what we believe would be the same contributions that come from the different poles in the loop formulation.

We can indeed consider the following two points of view, with respect to the vacuum for our problem:

1) We choose the "vacuum" to be the full 12 component bound state with an extra Higgs present, in a state with the spatial Higgs momentum distribution which we estimate couples to the annihilation. The initial state, consisting of the 12-constituent bound state without any extra Higgs, now has an energy below that of the vacuum, because of its *lacking* Higgs. That is to say the initial state has an energy $-E_{Higgs}$, where E_{Higgs} is the energy of the Higgs in the vacuum. So we think of this as the initiating $q\bar{q}$ pair having the initial energy $-E_{Higgs}$.

Now the process is that a $t\bar{t}$ pair annihilates to become a hole (really a double hole) in the vacuum, because the vacuum should have 12 constituents and after the annihilation there are only 10 left. So they really form a virtual s-channel hole. This hole represents that we have the 10-constituent bound state instead of the 12-constituent one. As we shall see in Appendix H, the mass difference between these bound states is $m_{10} - m_{12} \approx 950$ GeV. So the hole must be counted as having the energy 950 GeV. Since we start with a state with energy $-E_{Higgs}$, taken to be of the order $-m_t$ because the momentum is of that order, it means that the (double) hole must be strongly off shell.

In the relativistic notation we formally get a propagator with a denominator of the order of 950 GeV to the second power, which means that we assume it to be smaller than the corresponding object in the t channel by a factor $(950 \text{ GeV}/m_t)^2 = (950/172)^2 = 5.5^2 = 30.5$. But now, if we want to write the diagram in terms of the nonrelativistic vertex form (see Sec. 6.4), there is in this form a factor $1/\sqrt{2E_{hole}(\vec{p})}$ for each of the two vertices that must be extracted to get the vertex in the nonrelativistic formulation (52). Using the nonrelativistically normalized field $\phi_{hole\ nr}$, we would expect the transition matrix element between the 12-constituent bound

state and the 10-constituent one to be a rather simple overlap giving just unity in first approximation—of ignoring, for example, the difference in radii. Thus one of the two factors of 5.5 is used up by the factor $1/2E_{hole}$. So, as we think of varying the "big" number 5.5, we only get the s-channel contribution suppressed by one factor of 5.5. Since we assume that for the hole energy of the order of m_t only we would have gotten the same as in the t-channel case, this means that the suppression of the s-channel contribution is by the factor 5.5. However, we did not include the kinetic energy resulting from the spatial momentum being of the order of m_t, as given by Eq. (47). This means that the true suppression factor is rather $\sqrt{5.5^2 + 1} = 5.6$.

2) In this case we consider a vacuum which is simply the bound state with 10 constituents and we consider the Higgs to be the s-channel particle. Then the initial state has all the extra binding energy of the 12-constituent state compared to the 10-constituent one. That is to say now the initial energy is -950 GeV. In the Higgs propagator the factors in the denominator are of this order of magnitude, but we cannot absorb such strong suppression from even one of them by crudely identifying it with the factors $1/\sqrt{2E_{Higgs}}$ contained in the Higgs vertices, because the E_{Higgs} in the latter is given by the Higgs momentum and mass and these quantities in our model never reach more than about m_t. So, in this case 2, we indeed get a very small contribution only of order $1/5.5^2 = 1/30.5$ compared to the t channel or rather $1/5.6^2 = 1/31.4$, when we include the spatial momentum.

As we shall see in the following section, these two different tree-diagram estimates should really be *added*. In other words, the full s-channel contribution is suppressed, relative to the analogous t-channel term, by a suppression factor equal to the sum of the two above computed suppression factors:

$$\text{"suppression factor"} = 1/5.6 + 1/5.6^2 \approx \frac{1}{5.6(1 - 1/5.6)} \approx 1/4.6. \qquad (58)$$

Thus we shall calculate an s-channel contribution by first evaluating the coupling and combinatorial factors and then dividing the result by 4.6. We shall do this for the Higgs exchange in Sec. 6.7 and for the gluons in Sec. 6.8.

6.6 Arguing for adding the two terms

From the above discussion it may not be clear what we have to do with the two different results obtained under the points of view 1 and 2, respectively. We want to argue here that we should indeed add these two contributions. However, for this purpose, it is best to think of doing the calculation as a loop correction. Then we look at the correction to the binding energy as the result of the virtual split up of the 12-constituent bound state (the t ball) into a Higgs particle and the bound state consisting of only 10 constituents.

This means that it is truly a self-energy diagram in an effective field theory (with the various bound states as particles giving Feynman rules together with, e.g., the Higgs), which corrects the mass of the 12-constituent bound state.

When we formulate the mass correction in this loop way, we end up with a loop four-momentum q over which to integrate. Let us now think of the performance of the integral over the energy component q^0 of this loop four-momentum q: For fixed values of the spatial components of the loop four-momentum $\vec{q}$ the integrand is (basically) a product of two propagators, namely, one for the 10-constituent bound state and one for the Higgs. It therefore gets poles whenever one of these two particles is on shell. We imagine to approximate the whole loop integral by the sum over contributions from the neighborhood of these poles. Actually it is not difficult to see that, by an appropriate closing and deformation of the contour, you can prove that the loop integral over the q^0-dummy variable gives a sum over the pole residues (divided by 2π). Now the point is that these pole contributions can indeed be identified with the results from the formal tree diagrams just discussed under points 1 and 2. In fact the contribution from the Higgs-propagator pole (for positive Higgs energy) in the loop integrand gives us the formal tree diagram corresponding to the on-shell Higgs being considered part of the vacuum. The propagator in this formal tree diagram corresponds to the hole in the other 12-constituent part of the vacuum, so that it is really the propagator for the 10-constituent particle that lies under the hole. Thus this contribution from the pole of the Higgs propagator in the loop corresponds to case 1 above. Similarly the residue contribution from the pole of the 10-constituent bound state propagator in the loop integrand gives the contribution in which this 10-constituent bound state is identified with the vacuum. This is case 2 above.

Since we have now identified the two tree-diagram contributions from the previous section as being two contributions coming out of the same loop integral, we see that these contributions to shifting the mass of the 12-constituent bound state must be *added*.

6.7   Finding the s-channel Higgs correction to g_t

A certain quark in the bound state can only annihilate together with the antiquark having just the compensating color and spin. So there is among the antiquarks only one that can annihilate with a given quark into the Higgs. This means that the factor 11, corresponding to the number of quarks or antiquarks that can interact via t-channel exchange with a given quark, gets replaced by 1 for the s-channel exchange. In Sec. 2.1 we saw that, for u-channel exchange, we had to replace this factor of 11 by 5. So, by including the u channel, the interaction of a quark by Higgs exchange has a combined strength of coupling to the other constituents as if there were 16 of them coupling by only t-channel exchange. Thus, if the strength of the s-channel coupling, when allowed, had been just the same as

for the t channel, meaning just B, then the coupling strength of the s channel would have made up $\frac{1}{16}$ that of the t- plus u-channel Higgs exchange. Now these coupling strengths or scattering amplitudes are proportional to g_t^2. Thus, if the s-channel Higgs exchange results in a $1/16 = 6.25\%$ increase in the scattering amplitude, then we should decrease the previously predicted critical coupling $g_t|_{phase\ transition}$ by $\frac{1}{2} * \frac{1}{16} = 3.125\%$. But now, as we estimated above in Sec. 6.5, the s-channel propagator has to be suppressed by a factor of 4.6. This means then that, if we totally ignore the gluons, the percentwise decrease of the previously calculated critical $g_t|_{phase\ transition}$ would be $\frac{1}{32*4.6} = \frac{3.125}{4.6}\% = 0.68\%$. Since the gluon contribution to the potential does not depend on g_t^2, the correction of including the s-channel Higgs exchange will change the critical $g_t = g_t|_{phase\ transition}$ downward by 0.68%, i.e.,

$$\Delta \ln g_t|_{phase\ transition} = -0.68\% \text{ (from } s\text{-channel Higgs)}. \tag{59}$$

6.8   The gluon s-channel correction

Next we shall consider the change in the scattering amplitude, or equivalently the potential, from the exchange of gluons in the s channel. Each quark can interact by annihilating into a gluon with any one of the antiquarks, except when they form a color singlet together. We can take care of the latter exception by including a correction factor 8/9 in the scattering amplitude. Apart from this exception, we have interaction between all quarks with antiquarks, while neither quark and quark nor antiquark and antiquark can annihilate into gluons. So one quark can interact via the s channel with 6 antiquarks. Thus we can estimate the strength of the s-channel gluon exchange, counted in amplitude or potential, as being $6/16$ times as strong as the u- plus t-channel Higgs exchange, provided we replace the Higgs coupling $g_t^2/2 = (0.935)^2/2 = 0.437$ by the equivalent gluon coupling $e_{tt}^2 = 1.83$ [see Eq. (7)]. This replacement gives an increase in strength by a factor of 4.2. We must also remember to include the correction factor of 8/9. So finally we get the s-channel gluon exchange binding amplitude to be given relative to the combined u- and t-channel Higgs exchange by

$$\frac{\text{Gluon } s\text{-channel}}{\text{Higgs } t\text{-channel} + u\text{-channel}} = \frac{6}{16} * \frac{4.2}{4.6} * \frac{8}{9} = 0.304. \tag{60}$$

Here we also included the suppression factor of 4.6 from Sec. 6.5 for the s channel. Now, since the amplitude in which we calculated the correction is proportional to g_t^2, we obtain a backcorrection in g_t^2 of 30.4%, meaning that g_t^2 after the correction has to fill in the same as g_t^2 before with 30.4% subtracted, i.e., $g_t^2|_{corrected} = (1-30.4\%)g_t^2|_{before}$. So the correction to $g_t|_{phase\ transition}$ due to s-channel gluon exchange is downward by $-\ln(1 - 0.304)/2 = 18.1\%$, i.e.,

$$\Delta \ln g_t|_{phase\ transition} = -18.1\% \text{ (from } s\text{-channel gluons)}. \tag{61}$$

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6.9   s-channel summary

Summarizing we obtain

$$\Delta \ln g_t|_{phase\ transition} = -18.1\% - 0.68\% = -18.8\% \text{ (from full } s\text{-channel)} \qquad (62)$$

for the total correction from the s channel counted logarithmically.

7   Top mass field dependence corrections

If we could be allowed to use the masses m_i for the constituent particles undisturbed by the Higgs field having different values in different places in the interior of the bound state, then the expression (21) for the infinite momentum frame energy, derived in Appendix B, would lead to the expansion for m_{bound}^2 in Eq. (15) with $E_{binding}$ being just the nonrelativistic expression formally, even if this binding is big compared to the mass terms. In this sense the infinite momentum frame expansion justifies the formal nonrelativistic calculation, provided we take the former to mean the expansion of the mass squared being extrapolated without higher order terms.

Now, however, the masses occurring in this formula are supposedly changed, due to the average Higgs field in the interior of the bound state being smaller than in the outside. Such a change of the effective top-quark mass will naturally change the mass of the bound state and, at first, it looks like we should include a correction for this effect.

However, we see that in the approximation of the masses all being scaled by the same factor, due to the averaged Higgs field in the region where they are on the average, the whole bound state mass squared will simply be scaled by the square of this factor. This is simply a consequence of a dimensional argument, since the mass is the only dimensional quantity entering the calculation. The quantity p_z is, namely, only a formal going to infinity quantity.

Now, however, the quantity we are truly after is just the g_t value $g_t|_{phase\ transition}$ at which the mass squared of the bound state becomes zero. That is, however, a dimensionless quantity being asked for, and that *cannot depend on the value of the single mass scale quantity, the average mass.* Thus there should be no change in our phase transition coupling prediction due to such an effective mass change, provided we can count it as being by the same factor crudely all over the inside of the bound state. Thus actually, in the first approximation, no corrections are needed. This means 0% correction to first approximation.

7.1 Next order correction in effective mass variation with the field

Now, however, the approximation in which the effective mass inside the bound state should be just the same all over in space is not so terribly good. Rather we must take into account that, for a quark being in the deep interior of the bound state, the effective mass is smaller than for one being farther out in the outskirts of the bound state.

In order to correct for this *variation* of the effective mass, we imagine to have calculated the average mass m_{av} corresponding to the average Higgs field felt by the top quark. Then we may write the true space-dependent effective mass as

$$m(\vec{r}) = m_{av} + \Delta m(\vec{r}). \tag{63}$$

In Sec. 5 about the Higgs mass correction, we found that even at the center of the bound state the Higgs field was estimated to be $1/3$ of its faraway value, i.e., $v/3$, while on the average with respect to the constituent distribution it was $5v/9$. In the classical approximation the constituent can only reach out to the distance r where the kinetic energy becomes zero. Using the virial theorem, this corresponds to where the potential has fallen to numerically half the value at the average distance $< r >$. At this classical upper limit for the radial distance r, the field ϕ_h must be in the middle between the faraway value v and $5v/9$. Hence, at the classical boundary for the constituents, the Higgs field is $7v/9$. This already gives us an estimate of the fluctuation in the effective mass

$$\frac{|\Delta m|}{m_{av}} < \frac{5/9 - 1/3}{5/9} = \frac{2}{5} = 0.4 \tag{64}$$

or

$$\frac{|\Delta m|}{m_{av}} < \frac{7/9 - 5/9}{5/9} = \frac{2}{5} = 0.4. \tag{65}$$

A priori these coincident estimates are even overestimates and should be reduced by considering a flat interval distribution between $v/3$ and $7v/9$. Then, using $\frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} = 1/3$, we obtain a reduction factor of $1/\sqrt{3}$, which gives

$$\frac{\sqrt{< \Delta m^2 >}}{< m >} = \frac{|\Delta m|}{m_{av}} \approx \frac{5/9 - 1/3}{5/9 * \sqrt{3}} = \frac{2}{5\sqrt{3}} = 0.23. \tag{66}$$

7.2 An alternative mass fluctuation estimate

Another estimate of this "fluctuation" in the effective quark mass is gotten by using the fact that the relative spread in the radial distance is

$$\frac{\sqrt{< r^2 - < r >^2 >}}{< r >} = \frac{1}{\sqrt{3}}, \tag{67}$$

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which in turn implies a spread in the potential energy

$$\frac{\sqrt{(<V-<V>)^2}}{<V>} \approx \frac{1}{\sqrt{3}}. \tag{68}$$

Since $V \propto m_t - m(\vec{r})$, this means that

$$\frac{\sqrt{<(m_{av} - m(\vec{r}))^2>}}{m_t - <m(\vec{r})>} \approx \frac{1}{\sqrt{3}}, \tag{69}$$

and thus, using $m_{av} = \frac{5}{9}m_t$, we get

$$\frac{\sqrt{<\Delta m(\vec{r})^2>}}{m_{av}} \approx \frac{4}{5\sqrt{3}} = 0.46. \tag{70}$$

But here we did not take into account the flattening off of the potential in the center discussed in Appendix G, which we used in the first estimate.

Instead of arguing via first estimating the fluctuation in the distance and then calculating as if this fluctuation were small, we can directly calculate the fluctuation in $1/r$ or equivalently the Coulomb potential. In this case we get a value of $4/5$, which is $\sqrt{3}$ times bigger than (70).

7.3   Taylor expanding in the mass

In the nonrelativistic looking condition for the binding energy per particle just being equal to $m(\vec{r})/2$, given in Eq. (24), the only $m(\vec{r})$-dependent term with nonzero second derivative with respect to $m(\vec{r})$ is the kinetic energy term $\frac{\vec{p}^2}{2m}$. This term has the second derivative

$$\frac{\partial^2(\frac{\vec{p}^2}{2m})}{\partial m^2} = \frac{\vec{p}^2}{m^3}. \tag{71}$$

Provided that the average of the square of the $\Delta m(\vec{r})$ is as given by (66), i.e., $<(\Delta m)^2> = 0.23^2 * m_{av}^2 = 0.053 m_{av}^2$, we obtain an effective replacement for the kinetic term:

$$\frac{\vec{p}^2}{2m} \longrightarrow \frac{\vec{p}^2}{2m} + \frac{1}{2}\frac{\vec{p}^2}{m^3} * <(\Delta m)^2> = \frac{\vec{p}^2}{2m} + \frac{1}{2}\frac{\vec{p}^2}{m^3} * 0.053 m_{av}^2 = \frac{\vec{p}^2}{2m}(1 + 0.016) \tag{72}$$

7.4   Correction to $g_t|_{phase\ transition}$ from kinetic term mass fluctuation change

The change of the kinetic term effectively due to the mass *variation* by the factor $(1+0.016)$ means that, in Eq. (106) for the Hamiltonian in Appendix B, we have replaced the top-quark mass by a value $(1+0.016)^{-1}$ times as big. Thus the binding energy resulting from use of the modified version of this expression will, for dimensional reasons, be $(1+0.016)$

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times smaller than the usual Rydberg (3). To compensate for this decrease in the binding energy, the fourth power of g_t to which the Rydberg is proportional must be increased by 1.6%. Thus this correction, due to the variation of the effective mass over the bound state volume, to our critical Yukawa coupling prediction is that we increase the prediction by 0.4%.

Had we, instead of (66), used the alternative estimate (70) for the variation of the effective mass, we would have gotten a 4 times bigger value for $< (\Delta m)^2 >$. This would, in turn, mean an increase in the value for the predicted critical Yukawa coupling of $0.4\% * 4 = 1.6\%$. Had we used the even bigger estimate at the end of Sec. 7.2 for the fluctuation in the mass

$$\frac{\sqrt{< \Delta m(\vec{r})^2 >}}{m_{av}} \approx 4/5 = 0.80, \tag{73}$$

we would have gotten an increase of 4.8% in our predicted critical Yukawa coupling.

Since the bigger estimates of the correction correspond to using an unsmoothed potential even near the center of the bound state, they are probably less reliable. So we have a bit more confidence in the 0.4% estimate; but let us take $\Delta \ln g_t|_{phase\ transition} = 2\% \pm 3\%$ as a reasonable average.

8 Finite speed of Higgs exchange

In the non-relativistic calculations which we used, we took the interactions to be instantaneous and ignored the fact that the Higgs or gluon being exchanged between a couple of quarks or antiquarks after all only travels with the speed of light. Under such conditions the only Feynman digrams for t-channel exchange are the diagrams in which Higgses or gluons are exchanged one after the other. However, diagrams, in which a couple of quarks among our 12 interact by an exchange of two Higgses propagating simultaneously, are ignored in this approximation. By this we mean that a diagram in which the two Higgs propagators cross each other, when being exchanged, is what is ignored in the nonrelativistic approximation we used. We should however, to higher accuracy, include such possible effects of the emission and the absorption of the exchanged Higgs not being quite simultaneous.

We shall do this crudely here, by estimating the fluctuation caused by this effect in the distance r between the interacting quarks to be used in the potential (1): By the virial theorem, we have that the kinetic energy of a quark in its motion in the potential equals minus one-half of the potential energy and, thus, is just equal to the binding energy numerically. Since the binding energy, in the critical case which we look for, is $m_t/2$, we

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obtain on the average

$$< \frac{\vec{p}^2}{2m_t} >= m_t/2. \tag{74}$$

This implies that for a single component of the momentum—e.g., the component along the line connecting the interacting quarks—we have on the average

$$< p_x^2 >= m_t^2/3. \tag{75}$$

This implies a velocity component with a spread, due essentially to quantum fluctuations, of

$$< v_x^2 >= 1/3. \tag{76}$$

This in turn implies that the effective distance r to be used in the potentials such as (1) actually fluctuates by $\sqrt{1/3} * 100\% = 57\%$. Now the second derivatives of the potentials such as (1) are of the form

$$\frac{d^2 V_{t-channel\ Higgs}}{dr^2} = -\frac{2 * g_t^2/2}{4\pi r^3}. \tag{77}$$

Hence, by Taylor expanding the potential at the Higgs-delay corrected distance r_{dc} around the first or nonrelativistic approximation value r_{nr} for the distance between the interacting quarks, we get the fluctuation corrected effective potential to be:

$$\begin{aligned}
V_{eff}(r_{nr}) &= < V_{t-channel\ Higgs}(r_{nr}) + \frac{dV_{t-channel\ Higgs}(r_{nr})}{dr_{nr}}(r_{dc} - r_{nr}) && (78) \\
&\quad + \frac{1}{2}\frac{d^2 V_{t-channel\ Higgs}(r_{nr})}{dr_{nr}^2}(r_{dc} - r_{nr})^2 + ... > && (79) \\
&= V_{t-channel\ Higgs}(r_{nr}) + \frac{1}{2}\frac{d^2 V_{t-channel\ Higgs}(r_{nr})}{dr_{nr}^2}r_{nr}^2/3 + ... && (80) \\
&= (1 + 1/3)V_{t-channel\ Higgs}(r_{nr}) + ... && (81)
\end{aligned}$$

Thus, at the end, we get that this effect of the delay of the propagation of the (in first approximation) infinite speed Higgs exchange causes an effective spread in the distance r_{dc} to be used for evaluating the potential. This causes an effective increase in the potential by a factor of $1 + 1/3 = 4/3$, in our situation corresponding to the critical case of a zero mass bound state. In turn this means that the coupling g_t needed to provide this critical mass zero bound state should be corrected, by reducing it by the square root of this factor of $4/3$. This means that, instead of Eq. (18), we get the same equation but with the factor 0.0929 replaced by a number which is $(4/3)^2$ times bigger. So the equation now reads

$$0 = 1 - 0.0929 * (4/3)^2 * (0.456 + g_t^2)^2. \tag{82}$$

Thus we obtain the following value for the critical g_t, evaluated using the delay corrected effective potential (81):

$$g_t|_{phase\ transition} \simeq \sqrt{\sqrt{1/(0.0929 * (4/3)^2)} - 0.456} = 1.42. \tag{83}$$

Compared to the previous value for the critical g_t at this stage—before even the introduction of the b quark correction of Sec. 3—this is a downward correction given logarithmically percentwise by

$$\Delta \ln g_t|_{phase\ transition} = \ln \frac{1.42}{1.68} = -17.2\%.$$ (84)

9   Many body correction

Clearly the calculation made as if all the other quarks or antiquarks than the one considered were sitting in just one point cannot be correct; so we have in principle to make calculations on the system of the 12 constituents as a true many body system.

Here we shall do this in a rather crude way, only thinking of an ansatz in which the constituents are described by a factorizable wave function, meaning that it is a product of a wave function for each constituent independently of the other ones. Then it is obvious that the spread in the distance between a couple of constituents will be just $\sqrt{2}$ times bigger than that of the independent particle distributions. In turn this means that, to the extent that the expectation value of the momentum squared is given by—or at least varies as—the Heisenberg uncertainty relation, the independent $< \vec{p}^2 >$ will be twice that of the relative motion of one pair. This change will function as if the mass in the kinetic term were, for the many particle description, smaller by a factor 2 than in the starting relative position description. For dimensional reasons such a diminishing of the mass by a factor 2 would also diminish the resulting binding energy by this factor 2. In turn that would mean that we should correct our critical coupling upwards by a factor of $2^{1/4}$. This means a logarithmic percentwise increase of $100\% * \ln(2)/4 = 17.3\%$.

The many body corrections we are studying here reflect the fact that the calculation of the quark contributions to the binding, as if the individual pairs of top or antitop quarks could distribute themselves so as to minimize the energy of just that pair, cannot be quite correct. If two of the constituents are not essentially at the same site, it is impossible for a third one to be very close to both. In Appendix I we have illustrated this problem of the impossibility of having all the pairs have their optimized relative distance distribution, by using a Gaussian ansatz factorizable wave function for the whole bound state. Indeed the factor of 2 correction discussed in the previous section is essentially realized, but more precisely a factor of 2.16 is obtained (see Eq. (161)). This corresponds to an upward correction of $100\% * \ln(2.16)/4 = 19.3\%$ in our critical coupling $g_t|_{phase\ transition}$, which provides an upper bound for the many body correction considered. In fact one could a $priori$ very easily imagine that, by making a more complicated ansatz wave function, we could enhance the probability for the individual pairs having a small relative distance. In this way we could make the distribution between the constituents in a pair approach

more closely to the ideal ground state distribution for a two particle system. Certainly we must expect that the true wave function for the bound state system must have gone a bit in this direction compared to the ansatz in Appendix I.

It may be best to think about this effect, of somehow getting the wave function improved to cluster the constituents more on the short distances, as an antiscreening that could even be approximately described by a "dielectric" constant for the medium of constituents conceived as a material. With such a dielectric constant ϵ, the potential around a constituent is modified from the usual $\frac{g_t/\sqrt{2}}{4\pi r}$ form to $\frac{g_t/\sqrt{2}}{4\pi \epsilon r}$.

Let us now attempt to estimate an effective $1/\epsilon$ correction to use (on the potential) as a function of r. Such a correction factor $1/\epsilon$ would correct the quantity g_t^2 in our expression for the potential, which we might think of as being in an effective distance r dependent way.

All our earlier calculations before this section were made without any "many body" correction and assumed the absolutely most well-arranged relative distributions for all the pairs. However it is not possible to realize such a distribution for all the quarks simultaneously and thus these previous calculations provide an upper limit to the correction factor. It is therefore impossible that the correction factor $1/\epsilon$ could be more than a factor $\sqrt{2.16}$.

When we then think of the correction factor as dependent on an effective distance r, we must imagine a function of this r taking values between 1 and $\sqrt{2.16}$. It is clear that, for the distance r going to zero, it is hopeless to organize clustering and the correction factor must go to 1 there. Also at $r \to \infty$, where we think of a constituent isolated from the rest, there are essentially no particles to cluster with and no further clustering is possible. In practice the rest of the particles are already clustered in this case. So the further correction factor can only be 1 in this limit too. In the intermediate region in r, you would however expect some further clustering to take place compared to that of the ansatz wave function in Appendix I. So let us now assume that the correction factor as a function of r is reasonably smooth, say basically a second order polynomial, in the range of any significant population of the constituent distance r.

The maximal possible modification of our above correction of 19.3% could now only be achieved by having the maximal correction factor $1/\epsilon = \sqrt{2.16}$ around the typical or most likely distance, i.e., around $r = <r>$. But then the correction factor must also reach $1/\epsilon = 1$ as r goes to zero or to infinity effectively. Roughly this must mean that, for the tails of the distribution to both sides, we get the correction factor 1 rather than the $\sqrt{2.16}$. Let us very crudely estimate that, averaged over the distribution, we get the mean between the two values 1 and $\sqrt{2.16}$. That would mean that we would get g_t^2 replaced by $g_t^2 * (1 + \sqrt{2.16})/2$ or $g_t^2 * (2.16)^{1/4}$ using a geometrical mean instead. In order to

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compensate for that, we would need to decrease the critical coupling $g_t|_{phase\ transition}$ by a factor of $(2.16)^{1/8}$. This means a decrease of the critical g_t prediction from our model by 10.1%.

Together with the 19.3% increase, this "backcorrection" means that we would end up with a $19.3\% - 10.1\% = 9.2\%$ correction. It seems reasonable to consider this latter value, i.e., a 9.2% increase of $g_t|_{phase\ transition}$, as a lower bound for the many body correction. Therefore, crudely, we might present the result of this rather big many body correction as an increase of the predicted critical coupling by $(19.3+9.2)\%/2 \pm 9.2\%/2 = 14.2\% \pm 4.6\%$.

10 The SU(2) part of Z^0 and W exchange effects

We expect the effect of exchanging the time-components or rather Coulomb fields for Z^0, $W^\pm$ and the photon to be rather small, in as far as these exchanges are proportional to the fine structure constants for the $SU(2)$ and $U(1)$ gauge groups in the standard model, which are rather small. We should bear in mind that we already *have* included the scalar components of these *a priori* weak interaction gauge bosons. They were, namely, the so-called eaten Higgs exchanges, which were supposed to be larger because, as is explained in Appendixes D and E, the exchange of a scalar component becomes independent of the fine structure constants and is only given by the Yukawa coupling of the top quark.

So here we want to discuss, as a small correction, the inclusion of the timelike component exchanges of the weak gauge bosons. The exchange of a W boson has a similarity with the exchange of the eaten Higgs in that it converts a top quark into a bottom quark or oppositely. We could therefore roughly imagine that a timelike W exchange could—ignoring for the present what are left and what are right components of the quarks—take the place of an eaten charged Higgs component.

Very crudely we might therefore first simply imagine to include the W and Z^0 exchanges, by enhancing appropriately the eaten Higgs couplings analogous to the gauge particle time components in question. In the usual language, this means approximating the exchange due to the timelike components of the gauge particles by correcting by an overall factor the exchange due to the scalar components alone (which is the one we call the eaten Higgs exchange). Now the effective Coulomb potential for the eaten Higgs is

$$-\frac{g_t^2/2}{4\pi r}. \tag{85}$$

In the same notation the Coulomb potential corresponding to the exchange of W's between—now only left-handed—quarks becomes

$$-\frac{g_2\tau^a/2 * g_2\tau^a/2}{4\pi r} \text{ (only for left-handed quarks)}, \tag{86}$$

where, experimentally, we have at the m_Z scale

$$1/\alpha_2 = \frac{4\pi}{g_2^2} \approx 30. \tag{87}$$

Here we have crudely included W^0 exchange.

For a crude estimate we may take it that, in the roughly nonrelativistic situation, there should be equally many left-handed and right-handed top quarks, so that we can say in the eaten Higgs exchange case, we have to start a box loop with external right-handed top components. In analogy we have with the time components to start with left-handed top components, but that has approximately the same probability.

Using this way of arguing, we can effectively replace the τ^a matrices by the number 1. We then get that the ratio of the time-component exchange potential to the eaten Higgs exchange potential is given by the factor

$$\frac{1/30 * (1/2)^2}{(g_t^2/2)/(4\pi)} \approx \frac{1/120}{1/28} = 0.237 \tag{88}$$

Thus we may take a box loop, as discussed in Sec. 4, to have its two eaten Higgs propagators increased by the correction factor $1 + 0.237$. This would mean that, provided we had external top-quark states being guaranteed to be a certain linear combination of left-handed and right-handed components corresponding to that for nonrelativistic particles, the box diagram would be increased by a factor $(1 + 0.237)^2$. Hence the critical g_t, namely $g_t|_{phase\ transition}$, should be decreased by the fourth root of $(1 + 0.237)^2$, meaning percentwise a *decrease* by $\ln(1 + 0.237)/2 = 12\%$.

This is though an overestimate of the effect, because of the following "troubles":

1) Our estimate of getting the squared correction factor $(1 + 0.237)^2$ presupposes that interference terms, in the sense of box diagrams with one time component and one eaten Higgs in the same diagram, are really present.

2) There are not quite four W's corresponding to the, in total, four Higgses to be exchanged (as eaten Higgses or the original Higgs).

If we have to give up the interference term, the factor $(1 + 0.237)^2$ must be replaced by $1 + 0.237^2 = 1.056$, meaning only correcting the critical $g_t|_{phase\ transition}$ by *decreasing* it by $5.6\%/4 = 1.4\%$. The fact that we have only 3 W bosons rather than 4 means that we should reduce the ratio factor 0.237 from Eq. (88) to 3/4 of this number. That would alone bring the above 12% down to $3/4 * 12\% = 9\%$

So we take the correction coming from the exchange of the timelike components of the SU(2) gauge bosons to be between 9% and $3/4 * 1.4\% = 1.1\%$. In other words we take the correction to give a decrease of $g_t|_{phase\ transition}$ by $5\% \pm 4\%$.

In this crude estimate we really included the exchange of W^0, which corresponds to a superposition of Z^0 and the photon γ. Thus we are still left with having to include the orthogonal $U(1)$ superposition of Z^0 and γ in the next section.

11 U(1)-gauge boson exchange

The photon or better the $U(1)$-gauge boson exchange (a certain superposition of the photon and the Z^0 though mainly being the photon) may best be treated as effectively modifying the gluon coupling, since it couples similarly to the gluon.

The effective fine structure constant for the gluons, including the 4/3 from Eq. (7), is 0.109 *4/3 = 0.145 = 1/6.88, which is to be compared with the inverse fine structure constant for the $U(1)$ gauge group in the standard model $1/\alpha_1 \approx 100$ in the Z^0 mass region. This means that the potential from the $U(1)$-gauge boson exchange is down by a factor of 14.5 compared to that from the gluons. Since we found that the gluons make up about one-third of the potential for binding, we need only half of the 1/14.5 change in the g_t^2. In other words we must correct $g_t|_{phase\ transition}$ by a relative change of 1/2* 1/2 * 1/14.5 = 1/58. This means that the correction coming from the inclusion of the $U(1)$ gauge particle exchange causes the predicted critical g_t to be *decreased* by $1/58 = 1.72\%$

12 Renormalization Group scale discussion

The top-quark Yukawa coupling is strictly speaking a running coupling constant, and we should use its running value at the scale given by the typical momentum transferred by the Higgses, which are emitted in the scattering processes relevant inside the bound state. We have already found this to be m_t in Eq. (47). This typical momentum transfer is also crudely given by the inverse radius of the bound state which is, as already estimated in Appendix C, of the order of $(\sqrt{4/3}m_t)^{-1}$. That is to say that the critical Yukawa coupling $g_t|_{phase\ transition}$, which we estimate above, is to be interpreted as the running coupling at just the typical momentum, or by the radius given scale, $\mu \approx m_t$.

Usually the experimental result for the top-quark Yukawa coupling g_t is quoted as a running coupling at a scale of $\mu = m_t$, by making corrections to the measured pole mass. This gives the "experimental" value $g_t(\mu = m_t) = 0.935$, whereas a more naive extraction from the measured mass [4] of 172.6 GeV gives $g_t|_{naive} = 0.992$. The formula used to get the running mass m_t from the "naive" pole mass $M_t = g_t|_{naive} < \phi_h > /\sqrt{2}$ is [12]

$$m_t(M_t) = M_t \left[1 - 1.333 \frac{\alpha_s(M_t)}{\pi} - 9.125 \left(\frac{\alpha_s(M_t)}{\pi} \right)^2 \right]. \tag{89}$$

The effect described in this formula is that the top quark found experimentally is a "bare" top quark surrounded by some gluons.

Now the question is to what extent the top quarks in the bound state are also surrounded by gluons in the same way. Because from outside, at distances large compared to its radius, the total bound state is seen as a colorless particle, there must be such a

destructive interference between the gluons from the different quarks or antiquarks that there will be no gluons at distances much bigger than the radius. But that means that there are to first approximation no gluons surrounding the quarks, when they are inside the bound state. Thus the bound quarks are, from the viewpoint of Eq. (89) the "bare" ones, described by the running mass. This is the reason that we shall, in first approximation, compare our prediction to $g_t(\mu = m_t) = 0.935$ rather than to the naive value $g_t|_{naive} = 0.992$.

By accident the scales associated with our critical coupling $g_t|_{phase\ transition}$ and the experimental running mass are essentially the same. So we do not need to make any renormalization group correction. Nonetheless there is an ambiguity in defining the precise scale and we will take a typical uncertainty in the definition to be a factor of square root of 2. In order to calculate the change Δg_t in the top-quark coupling generated by a shift in the scale μ by a factor $\sqrt{2}$, we need to use the β-function

$$\frac{dg_t}{d\ln\mu} = \frac{g_t}{16\pi^2}\left(\frac{9}{2}g_t^2 - 8g_3^2 - \frac{9}{4}g_2^2 - \frac{17}{12}g_1^2\right).$$ (90)

Here $g_3 = g_s$, g_2 and g_1 are the $SU(3) \times SU(2) \times U(1)$ running gauge coupling constants, related to their associated fine structure constants by $\alpha_i = g_i^2/(4\pi)$.

Using the experimental values $g_t = 0.935$, $g_3^2 = 4\pi * 0.109$, $g_2^2 = 4\pi/30$ and $g_1^2 = 4\pi/100$ and taking $\Delta\ln\mu = \ln(2)/2 = 0.347$, we get that $\Delta\ln g_t = 0.347 * (3.9340 - 10.958 - 0.942 - 0.178)/(16\pi^2) = 1.79\%$. Rounding this off to 2%, we claim that the result for the correction due to the renormalization group running scale is just $0\% \pm 2\%$.

13 Collecting results.

It is the value $g_t|_{phase\ transition} = 1.19$ from Eq. (25) that has to be changed by the total correction factor, resulting from all the corrections to eq. (25) discussed throughout the paper and presented in Table 1. The collected "total" percentwise logarithmic correction turns out to be $-17.3\% \pm 14.2\%$. Thus the running Yukawa coupling for the top quark at the m_t scale is predicted, under our basic assumption that the mass of the bound state shall be just tuned in—mysteriously—to be small, to be

$$g_t = g_t|_{phase\ transition} = \exp(-17.3\%) * 1.19 = 1.001 \pm 14.2\% = 1.00 \pm 0.14$$ (91)

This result is to be compared with the value from experiment [4], obtained from a top-quark pole mass of 172.6 ± 1.4 GeV:

$$g_t(m_t) = 0.935 \pm 0.008.$$ (92)

Name of correction	Section	Logarithmic %	Estimated theoretical uncertainty in %
Adjustment eaten	4	4.0	8.0
Higgs mass	5	5.2	4.0
s-channel	6	-18.8	6.0
m_t field dependence	7	2.0	3.0
Finite speed	8	-17.2	6.0
Many body	9	14.2	4.6
Z^0 and W exchange	10	-5.0	4.0
$U(1)$-gauge exchange	11	-1.7	0.6
Renorm group	12	0.0	2.0
Total		-17.3	14.2

Table 1: Collecting corrections.

This means that our prediction from the masslessness of the bound state is fulfilled up to $(1.001 - 0.935)/0.142 = 0.46$ standard deviations.

At least this calculation means that the very exotic bound state we propose has a mass squared which is down by a factor of the order of 0.142 relative to the natural mass squared scale for this type of bound state, namely, the mass squared of $12m_t$, i.e. $144m_t^2 \approx 4 \text{ TeV}^2$. That is to say that the mass of the bound state must be at least as small as of the order $12m_t * \sqrt{4 * 0.142} = 1560 \text{ GeV}$. However, the real point is that it could easily within the errors be much lighter, e.g., zero mass. Looked upon as such an estimate of the bound state mass, our calculation at first sight appears to not be so impressive. However, it means that if the top-quark Yukawa coupling g_t deviated outside our error estimate, it would be likely that either (i) there would be such strong binding that a condensate would unavoidably have formed and we would live in a phase with such a condensate safely dominating, or (ii) the binding would be very tiny compared to $12m_t$. Essentially the case of binding with a binding energy just of the order of $12m_t$ is what corresponds, in the g_t formulation, to a rather narrow and impressive range in which the experimental coupling quite remarkably lies.

The really remarkable thing coming out of our calculation is that, by requiring the masslessness for the bound state, we get the empirical Yukawa coupling with such a high accuracy of 14.2% that there is a rather striking agreement. In itself it is remarkable even to get the right order of magnitude for the Yukawa coupling. However the fact that we get it just right with a 14.2% accuracy is something that would only occur, even if the agreement within a factor e were already guaranteed, in one out of 8 cases. So it almost calls for some underlying theory to explain that coincidence. We would say that the "multiple point principle" of requiring many vacua with the same energy density [1, 6]

would function as such a theory, if we, namely, take there to be two vacua, one with and one without a bose condensate of the bound state discussed in this article. We can say it is a case of a strange fine-tuning of the Yukawa coupling for the top quark and that a fine-tuning machinery is called for.

14 Conclusions

The main content of the present article has been a calculation, performed inside the standard model, of the mass of a special bound state. In fact we calculated the mass of the bound state formed from 6 top quarks and 6 antitop quarks. The importance of just this set of quarks and antiquarks is that they form a closed shell, so that there is a significant decrease in the strength of binding when the next quark or antiquark is added to the system. The remarkable result we found is that the top-quark Yukawa coupling experimentally has just the value that allows this bound state of the $6t + 6\bar{t}$ to be totally *massless*. That is to say, within the uncertainty, it can very easily be that the binding is so strong as to just cancel the mass energy of the constituents. In fact we formulated our calculation so as to evaluate just that specific value of the top-quark Yukawa coupling, which gives precisely zero mass for the bound state of its 12 constituents. It must be admitted that this conclusion of the 6 top and 6 anti-top quark state even binding—let alone so strongly as to get zero mass—is at variance with the conclusion of Kuchiev, Flambaum and Shuryak [5] who do not even have it bind. But we have included several further important effects—such as eaten Higgses and corrections to the Higgs mass to be used *inside* the bound state—in our calculation of the binding strength. As suggested by a toy model calculation in Appendix J, there is reason to believe that the mass of the bound state—including the question of binding—has a kink behavior as a function of, e.g., the Yukawa coupling g_t. So two calculations performed on different sides of such a kink value of the variable g_t could *a priori* give quite different results. Depending on where exactly "the phase transition" is only one of two such calculations would be correct, the other one being analogous to calculating the properties of fluid superheated water for a temperature where the true phase is the vapor phase.

So we can suspect that, provided one included enough of our above-mentioned corrections, the correct "phase" for the calculation would be the one with a "collapsed" Higgs field inside the hoped for bound state. However Kuchiev et al. [5] made their calculation in the phase with an uncollapsed Higgs field, i.e., the calculation is in the wrong phase. But presumably, without the inclusion of eaten Higgs exchange and our other corrections, the experimental value of g_t would lie in the phase in which Kuchiev et al. worked.

Our aim was then to see to what accuracy the rather mysterious coincidence of bound

state masslessness actually works in the phase with a collapsed Higgs field. So we wanted to compute the critical Yukawa coupling $g_t|_{phase\ transition}$, defined here as the one making the bound state massless, as accurately as possible. We did that by first calculating it in a rather crude way, leading to the value $g_t|_{phase\ transition} = 1.19$ in Eq. (25). In this first step in the calculation we included both Higgs exchange and gluon exchange in the t channel and the u channel, but did not yet include the s-channel exchange (which is more difficult to calculate); we also very crudely corrected for the fact that there could also be left-handed b quarks and antiquarks virtually present in the bound state, essentially replacing the t quarks from time to time. We also, in this first calculation, used a very crude approximation of letting each quark encircle a conglomerate of all the other 11 quarks concentrated into one point. However we did take the double counting into account and thus really calculated with only 11/2 particles in the center. We made the calculation totally nonrelativistically, as is almost needed to calculate a bound state without having to truly go to the Bethe-Salpeter equation.

After this first calculation, we then made a series of 9 corrections listed in Table 1 in the foregoing section. Together these corrections led to lowering the predicted critical Yukawa coupling by 17.3% counted logarithmically. The resulting Yukawa coupling that would give just zero mass for the bound state of $6t + 6\bar{t}$ was thus computed to be $g_t|_{phase\ transition} = 1.00 \pm 0.14$. This uncertainty of 14% is only a very crude estimate of the uncertainties in the many corrections added up in quadrature. At first sight this 14% uncertainty appears to be a small error. However, one should bear in mind that what we really estimate is g_t^4 rather than g_t itself. Therefore the true uncertainty on our calculation, namely, for g_t^4, is in fact rather of the order of 70%. We performed the calculation so as to estimate the running Yukawa coupling at the m_t mass scale, where the pole mass correction performed on the experimentally measured top-quark mass leads to the experimental running Yukawa coupling value $g_t(\mu = m_t) = 0.935 \pm 0.008$. This experimental value has thus fallen, within an uncertainty of only 0.46 standard deviations, on the value needed to make the bound state massless.

Acknowledgements

Appendix A: Notation

In Sec. 2 we work with just one Hermitian ("real") field for the physical Higgs particle ϕ_h. Here we shall give some notation for this field and the related complex doublet field ϕ_H.

We take the Lagrangian for the real field ϕ_h to be normalized as

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu \phi_h)^2 + \frac{1}{2}|m_{hb}|^2 \phi_h^2 - \frac{\lambda}{8}\phi_h^4 + \frac{g_t}{\sqrt{2}}\overline{\psi}_t \psi_t \phi_h + \overline{\psi}_t \gamma^\mu \partial_\mu \psi_t + ... \tag{93}$$

Phenomenologically we know that the vacuum expectation value of the Higgs field is

$$< \phi_h > = |m_{hb}|/\sqrt{\lambda/2} = v = 246 \text{ GeV}, \tag{94}$$

while the physical Higgs mass becomes

$$m_h = \sqrt{2}|m_{hb}| = \sqrt{\lambda}\, v. \tag{95}$$

So, for example, for a Higgs mass of $m_h = 115$ GeV, we find in this notation that $\lambda = \frac{(115 \ GeV)^2}{(246 \ GeV)^2} = 0.218$

In order to treat the eaten Higgses too, as we do in Sec. 3, we must introduce the Higgs doublet complex field notation in which

$$\phi_H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{96}$$

where then we take

$$\phi^0 = \frac{1}{\sqrt{2}}(\phi_h + i\phi_2), \tag{97}$$

with ϕ_h and ϕ_2 real. With this relation we are then forced to take the Lagrangian density for the complex field doublet to be

$$\mathcal{L}_\mathcal{H} = |D_\mu \phi_H|^2 + |m_{hb}|^2 \phi_H^\dagger \phi_H - \frac{\lambda}{2}(\phi_H^\dagger \phi_H)^2. \tag{98}$$

With the substitution (97), the Yukawa interaction term in (93) becomes

$$\mathcal{L}_\mathcal{H} = ... + g_t(\overline{\psi}_{tR}\phi_H^\dagger \psi_{tbL} + h.c.) + ... \tag{99}$$

Here we have introduced the splitting of the Dirac spinor into its Weyl representation components—meaning left and right being considered separately—and also introduced the left-handed b field, so that we have now a doublet of left-handed fields under the weak isospin:

$$\psi_{tbL} = \begin{pmatrix} \psi_{tL} \\ \psi_{bL} \end{pmatrix}. \tag{100}$$

We also denote the right-handed components of the t field by ψ_{tR}.

Appendix B: Infinite momentum frame for nonrelativistic approximation and analyticity

We shall here see how a nonrelativistic atomlike theory gets written in the infinite momentum frame. Let us consider a cluster of n constituent particles numbered by $i = 1, ..., n$ with masses m_i and longitudinal momenta p_{zi} written as

$$p_{zi} = x_i p_z. \tag{101}$$

Here p_z is some very large momentum used to specify the very fast moving frame that is the IMF. Then the energy of the cluster of particles in this frame, in which we think and in which the particles move very fast, is expanded as follows

$$E_{IMF\ cluster} = p_z + \left(\sum_{i=1}^{n} \frac{m_i^2 + \vec{p}_{Ti}^2}{2x_i} \right) / p_z + \frac{1}{2} \sum_{i,j,\ i \neq j} V_{ij}/\gamma_{ij}. \tag{102}$$

We use the notation $\vec{p}_{Ti}$ for the transverse part of the momentum of particle i and $\vec{p}_T = \sum_{i=1}^{n} \vec{p}_{Ti} = 0$. Here the nonrelativistic scalar potential V_{ij}, for particle i influencing particle j, is being boosted from the cluster rest frame to the infinite momentum frame and thereby Lorentz contracted. Because of the Lorentz contraction of the wave function for particle j, its scalar interaction goes down by the factor $1/\gamma_j = \sqrt{1 - v_{lj}^2}$, where v_{lj} is the longitudinal velocity of particle j. If we had thought about the interaction the opposite way around, we would have gotten $1/\gamma_i$ instead. But if the particles keep interacting they must run with the same speed and that would mean $\gamma_i \approx \gamma_j$, so that we can put γ_{ij} equal to both of them.

Since the longitudinal $\gamma_i = p_z x_i / m_i$ in the infinite momentum limit, we have in this case of the same longitudinal velocity that $x_i/x_j = m_i/m_j$.

It is also obvious that (101) implies the well-known normalization

$$\sum_{i=1}^{n} x_i = 1. \tag{103}$$

Especially in the first nonrelativistic approximation for the internal motion of the cluster, the relative velocities are small and thus the x_i's are proportional to the corresponding m_i's. Also, in this first approximation of $x_i \propto m_i$, we could for instance write, using the average of $1/\gamma_i$ and $1/\gamma_j$ for $1/\gamma_{ij}$:

$$E_{IMF\ cluster} = p_z + \left(\sum_{i=1}^{n} \frac{m_i^2 + \vec{p}_{Ti}^2}{2x_i} + \frac{1}{2} \sum_{i,j,\ i \neq j} V_{ij} * \frac{1}{2}(\frac{m_i}{x_i} + \frac{m_j}{x_j}) \right) / p_z \tag{104}$$

$$= p_z + \frac{1}{2p_z} \left(\sum_{i=1}^{n} \left(\frac{m_i^2}{x_i} + 2\frac{\vec{p}_{Ti}^2}{2x_i} \right) + \frac{1}{2} \sum_{i,j,\ i \neq j} V_{ij} * (\frac{m_i}{x_i} + \frac{m_j}{x_j}) \right). \tag{105}$$

Comparing with the IMF expansion (13) of E_{IMF}, we see that the squares of the eigenmasses for the bound states in the channel considered are given as eigenvalues of the operator $\left(\sum_{i=1}^n \left(\frac{m_i^2}{x_i} + 2\frac{\vec{p}_{Ti}^2}{2x_i} \right) + \frac{1}{2} \sum_{i,j,\ i\neq j} V_{ij} * \left(\frac{m_i}{x_i} + \frac{m_j}{x_j} \right) \right)$, so that we determine the bound state masses from the eigenvalue equation:

$$\left(\sum_{i=1}^n \left(\frac{m_i^2}{x_i} + 2\frac{\vec{p}_{Ti}^2}{2x_i} \right) + \frac{1}{2} \sum_{i,j,\ i\neq j} V_{ij} * \left(\frac{m_i}{x_i} + \frac{m_j}{x_j} \right) \right) \Psi = m_{bound}^2 \Psi. \tag{106}$$

The remarkable thing for us here is that there is no obvious reason why this eigenvalue equation should have any singular behavior for m_{bound}^2 at zero. Therefore we expect that the eigenvalues, meaning the masses squared of the bound states, will behave smoothly as a function of the parameters such as g_t. That suggests confidence in using a low order Taylor expansion in the parameters, even when the bound state mass squared m_{bound}^2 comes close to zero. In other words we expect to have no singularities at $m_{bound}^2 = 0$, when we use the eigenvalue equation (106) to obtain the IMF-mass squared of the bound state.

In order to check that we do indeed get to the slightly surprising factor $1/2$ in (20), meaning that the binding energy in the formal nonrelativistic calculation should only compensate one-half of the mass in order to make the bound state just massless, we shall here take the nonrelativistic approximation to our IMF formalism: With the nonrelativistic approximation in mind, in the frame of the bound state, we define the Δx_i's by

$$x_i = \frac{m_i}{\sum_j m_j} + \Delta x_i \equiv x_{i\ old} + \Delta x_i. \tag{107}$$

Below we shall prove that, by Taylor expanding the term $\frac{m_i^2}{x_i}$ in the expression (105) for $E_{IMF/cluster}$, we obtain the longitudinal part of the kinetic energy quite analogous to the transverse part already present.

Neglecting the Δx_i and inserting $x_i = x_{i\ old} = \frac{m_i}{\sum_j m_j}$ into (105), we get

$$
\begin{aligned}
E_{IMF\ cluster} &= p_z + \frac{1}{2p_z} \left((\sum_{i=1}^n m_i)^2 + 2(\sum_k m_k) * \left(\sum_i \frac{\vec{p}_{Ti}^2}{2m_i} + \frac{1}{2} \sum_{i,j,\ i\neq j} V_{ij} \right) \right) &(108) \\
&= p_z + \frac{\sum_j m_j}{2p_z} \left((\sum_k m_k) + 2 \left(\sum_i \frac{\vec{p}_{Ti}^2}{2m_i} + \frac{1}{2} \sum_{i,j,\ i\neq j} V_{ij} \right) \right) &(109) \\
&= p_z + \frac{1}{2p_z} \sum_{i=1}^n \frac{m_i}{x_{i\ old}} (m_i + 2H_i|_\perp), &(110)
\end{aligned}
$$

where

$$H_i|_\perp = \frac{\vec{p}_{Ti}^2}{2m_i} + \frac{1}{2} \sum_{j,\ i\neq j} V_{ij}. \tag{111}$$

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Actually we will now show that the Taylor expansion of the main term $\left(\sum_i \frac{m_i^2}{x_i}\right)/(2p_z)$ in (105) has a dependence on the longitudinal momentum of the constituents, which can be interpreted as the missing longitudinal momentum dependent part of the kinetic energy, looking quite analogous to the transverse part.

In the "at rest" limit, in which the particles in the cluster lie still in the cluster rest frame, we have $p_{zi} = x_i * p_z$ with $x_i = x_{i\ old} = m_i/(\sum_j m_j)$. However, if the particles are not at relative rest, the x_i's will deviate from the $x_{i\ old}$ as in (107):

$$x_i = x_{i\ old} + \Delta x_i. \tag{112}$$

Of course

$$\Delta x_i = \frac{\Delta p_{zi}}{p_z}, \tag{113}$$

where Δp_{zi} stands for the deviation of the longitudinal momentum of the ith particle from the value $x_{i\ old} p_z = \frac{m_i p_z}{\sum_j m_j}$, which is the momentum it would have in the "resting" approximation. This Δp_{zi} could roughly be considered to be the result of the boosting of the longitudinal component of momentum $p_{nr\ i}$ of the particle i, measured in the rest frame of the cluster, from the rest frame of the cluster to the infinite momentum frame we consider. In the nonrelativistic approximation, in which the velocity of this cluster rest frame is given by the velocity v with associated γ satisfying $v\gamma = p_z/(\sum_j m_j)$ (or approximately for very large p_z just $\gamma = p_z/(\sum_j m_j)$), we have

$$\Delta p_{zi} = \gamma p_{nr\ i} = \frac{p_z}{\sum_j m_j} p_{nr\ i}. \tag{114}$$

The Taylor expansion of the main x_i-dependent term in the $E_{IMF\ cluster}$ now gives

$$\frac{1}{2p_z}\frac{m_i^2}{x_i} = \frac{1}{2p_z}\left(\frac{m_i^2}{x_{i\ old}} - \frac{m_i^2}{x_{i\ old}^2}\Delta x_i + \frac{m_i^2}{x_{i\ old}^3}\Delta x_i^2 + ...\right). \tag{115}$$

With the insertion of (114) into the third term in this expansion, which is proportional to Δx_i^2, we get

$$\frac{1}{2p_z}\frac{m_i^2}{x_{i\ old}^3}\Delta x_i^2 = \frac{1}{2p_z}\frac{m_i^2}{x_{i\ old}^3}\frac{\Delta p_{zi}^2}{p_z^2} \tag{116}$$

$$= \frac{1}{2p_z^3}\frac{(\sum_j m_j)^3}{m_i}\Delta p_{zi}^2 \tag{117}$$

$$\approx \frac{p_{nr\ i}^2}{2p_z x_{i\ old}}. \tag{118}$$

We see that this term (118) is precisely analogous to the transverse term $\frac{p_{Ti}^2}{2p_z x_i}$. The second term in the Taylor expansion, the one going linearly in Δx_i, is quickly seen to be proportional to the sum $\sum_i \Delta x_i$ which is zero, provided one keeps to the normalization $\sum_i x_i = 1$.

Now, adding these terms proportional to Δx_i^2 to (110), we obtain

$$E_{IMF\ cluster} = p_z + \frac{1}{2p_z} \sum_{i=1}^{n} \frac{m_i}{x_{i\ old}}(m_i + 2H_i), \tag{119}$$

where

$$H_i = \frac{\vec{p}_i^2}{2m_i} + \frac{1}{2} \sum_{j,\ i \neq j} V_{ij}. \tag{120}$$

Here $\vec{p}_i^2 = \vec{p}_{Ti}^2 + p_{nr\ i}^2$ is the total momentum squared of particle i in the cluster rest frame.

Appendix C: Radius estimate

In order to estimate the radius of our bound state in the critical coupling case, we may use Eq. (24) and the virial theorem. From the virial theorem for a $1/r$ potential, it follows that the total binding energy comes about by the average of the potential energy making up twice the binding energy (being negative like the binding energy), while the kinetic energy is numerically equal to the binding energy but is positive and thus compensates away one-half of the potential energy. Now, according to (24), the binding energy per constituent particle must be $m_t/2$ in the critical case. It therefore follows, from the above virial theorem consideration, that we must have

$$m_t/2 = <T> = <\frac{\vec{p}^2}{2m_t}> = \frac{<p_x^2> + <p_y^2> + <p_z^2>}{2m_t}. \tag{121}$$

For symmetry reasons it then follows that

$$<p_x^2> = <p_y^2> = <p_z^2> = \frac{m_t^2}{3}. \tag{122}$$

Now we want to use the fact that, in the ground state, the Heisenberg uncertainty relation

$$<x^2><p_x^2> \geq 1/4 \tag{123}$$

is actually an approximate equality, so that we really have

$$<x^2><p_x^2> \approx 1/4. \tag{124}$$

The ground state of a system like ours, or an atom, is achieved by concentrating the constituents with minimal energy as closely together as the Heisenberg uncertainty relation allows. Now the true equality is achieved only for a Gaussian wave function. However the deviation from Gaussian form only comes in to second order (in some parameter measuring the deviation from the Gaussian form of the wave function) because, imagining an abstract Taylor expansion for the deviation from the Heisenberg uncertainty relation, it could only have second order terms without violating the inequality.

Inserting $< p_x^2 >= m_t^2/3$ from (122) into (124), we get in the ground state in the critical case

$$< x^2 >=< y^2 >=< z^2 >\approx \frac{3}{4m_t^2}. \tag{125}$$

From here

$$< r^2 >= 3 < x^2 >\approx \frac{9}{4m_t^2}. \tag{126}$$

With the wave function $\psi \propto \exp(-r/r_0)$, one easily finds

$$< r^2 >= 3r_0^2 \tag{127}$$

and so derives that

$$r_0 \approx \sqrt{3/4}\frac{1}{m_t} \tag{128}$$

from (126) and (127).

Note that this argument is true independent of whether we have gluon or Higgs exchange or a mixture, provided the exchanged particle is sufficiently light so that the scaling properties assumed about the potential, when using the virial theorem, remain valid.

Appendix D: Eaten Higgses from W and Z

We shall explain how the nonconserved part of the current, coupling to W, causes a propagator to be inversely proportional to the gauge coupling squared for small four-momentum transfer q^2 and which, thus, can cancel the squared gauge coupling coming from the vertices. In this way we can, even in the limit of the gauge coupling going to zero, have a nonzero exchange force due to the gauge particles W and Z^0.

When, for instance, a top quark is converted into a bottom quark by emission of a W, the transition current $j_{W^+}^\mu = \frac{g_2}{\sqrt{2}}\overline{\psi}_{bL}\gamma^\mu\psi_{tL}$ is not conserved due to the masses of the top and bottom quarks. [Here we took the general W current to be $j^{a\,\mu} = \frac{g_2}{2}\overline{\psi}_{btL}\gamma^\mu\tau^a\psi_{btL}$ and the normalization $W^+ = \frac{1}{\sqrt{2}}(W_1 + iW_2)$, so that W^+ will couple to the current $j_{W^+}^\mu = \frac{g_2}{\sqrt{2}}\overline{\psi}_{bL}\gamma^\mu\psi_{tL}$.] In fact, using the equations of motion for the quarks in the background of the Higgs field, the divergence of the current becomes

$$\partial_\mu j_{W^+}^\mu = \partial_\mu j^\mu = -i\frac{g_2}{\sqrt{2}}m_t\overline{\psi}_{bL}\psi_{tR} + i\frac{g_2}{\sqrt{2}}m_b\overline{\psi}_{bR}\psi_{tL} \neq 0. \tag{129}$$

Here m_t and m_b are the top and bottom quark masses, which are given by

$$m_t = \frac{g_t}{\sqrt{2}} < \phi_h >; \quad m_b = \frac{g_b}{\sqrt{2}} < \phi_h > . \tag{130}$$

(If we consider the b quark massless as a good approximation, then $g_b = 0$ and $m_b = 0$.)

Considering the inverse propagator for the gauge boson, as obtained from $-1/4 * F_{\mu\nu}^i F^{i\,\mu\nu} - m_W^2 A_\mu^i A^{i\,\mu}$, we see that the kinetic part of this inverse propagator

can be zero for currents having the direction of a four gradient (as is a consequence of the gauge invariance of this kinetic part). Thus the propagator goes as the inverse of m_W^2 in such cases. But now the W only got its mass m_W nonzero due to the Higgs field and this mass is actually proportional to the gauge coupling $m_W = \frac{g_2}{2} < \phi_h > = \frac{g_2}{\sqrt{2}} < \phi_H >$. Thus the propagator for (nonconserved) currents not coupling to the kinetic part of the inverse propagator becomes proportional to the inverse gauge coupling constant squared $\propto 1/g_2^2$. The exchange amplitude for this nonconserved contribution thus has its g_2 dependence canceled. This then means that, even in the limit of the gauge coupling $g_2 \to 0$, the exchange of the massive gauge boson cannot be ignored when the current is not conserved. In this limit the exchange amplitude can only depend on the other coupling constant, the Yukawa coupling, and indeed it is physically really just the exchange of the eaten Higgs components that comes out of this limit from the gauge particle exchange.

The conclusion we want to draw in this Appendix is this: In the formalism in which one considers massive gauge bosons, such as W and Z^0, one only has to consider, in addition, the *physical* Higgs particle components. However, in the limit of letting the gauge couplings g_2 and g_1 go to zero, the gauge boson exchange *does not fully decouple*. Rather, in this limit, the gauge boson exchange simply becomes what one would get, in addition to the physical ϕ_h Higgs exchange contribution, by including the full Higgs field ϕ_H with all its four real components. So in this limit one is truly led to the pure Higgs model, *but with all the components, including the previously eaten ones.*

Although the above argument was very suggestive of what likely goes on in the limit of very weak gauge couplings, we actually should check that we do obtain the correct exchange amplitude corresponding to the eaten Higgs. A simple check of this can be done as follows, if it is accepted that we can be allowed to talk about the nonconserved part of the current and take it to be in momentum representation:

$$j_{W+}^\mu|_{nonconserved} = j^\mu|_{nonconserved} \quad = \quad \frac{1}{q^2} q^\mu q_\nu j^\nu \tag{131}$$

$$= \frac{1}{q^2} q^\mu \frac{1}{i} \partial_\nu j^\nu \quad = \quad \frac{1}{q^2} q^\mu (-\frac{g_2}{\sqrt{2}} m_t \overline{\psi}_{bL} \psi_{tR} + \frac{g_2}{\sqrt{2}} m_b \overline{\psi}_{bR} \psi_{tL}). \tag{132}$$

Using this "only nonconserved part of the current" together with a W propagator put equal to just $\frac{1}{m_W^2}$, as is expected to be a good approximation for $g_2 \to 0$, we can formally obtain an expression for the amplitude corresponding to the W-exchange diagram for the scattering of a pair of quarks. For the quark transitions from t to b and oppositely from b to t at the two vertices and putting $m_b = 0$ for simplicity, we get the following expression:

$$\frac{j_{nonconserved}^\mu j_{\mu\ nonconserved}^\dagger}{m_W^2} \quad \approx \quad (g_2/\sqrt{2})^2 m_t \overline{\psi}_{bL} \psi_{tR} * m_t \overline{\psi}_{tR} \psi_{bL} * \frac{1}{q^2 m_W^2} \tag{133}$$

$$\approx \quad g_t \overline{\psi}_{bL} \psi_{tR} * \frac{1}{q^2} g_t \overline{\psi}_{tR} \psi_{bL}. \tag{134}$$

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This scattering amplitude for the quark transitions is precisely what you get by exchange of an "eaten" Higgs.

Appendix E: Counting eaten Higgses

In Sec. 3 we introduced the extra Higgs components, which are eaten by the gauge particles, together with the b quark. They were used to consider (formally) the contribution of a box diagram to the elastic scattering of the weak singlet right-handed top quark t_R and its antiparticle $\overline{t_R}$. Because of the fact that there were now 4 real components of the complex doublet Higgs field ϕ_H propagating in the loop rather than just the one real physical Higgs field ϕ_h considered in Sec. 2, we argued that the scattering amplitude must increase by a factor of 4. In this Appendix we shall now confirm this factor of 4, by simply evaluating the ratio of box-diagram amplitudes for $t_R\overline{t_R}$ scattering in the two cases: (i) including only the physical Higgs component ϕ_h and the left-handed top quark t_L in the loop, and (ii) including all 4 components of the Higgs field ϕ_H and both the left-handed top t_L and bottom b_L quarks in the loop.[8]

Using the notation of Appendix A, we see that the box diagram—with two left-handed top or bottom quarks and two Higgses in the four-sided loop—gets changed in the following ways, when going from case (i) including only the physical Higgs component to case (ii) including the full 4 component Higgs field of the standard model:

1) We get rid of the $1/\sqrt{2}$ in the Yukawa coupling Lagrangian density. This means that the scattering amplitude goes up by a factor of $\sqrt{2}^4$ relative to case (i) with only the physical Higgs.

2) After including all four components, we have to evaluate an SU(2) trace for the box diagram, corresponding to the fact that a weak isodoublet circles around the box loop. This means that the amplitude goes up by a factor of 2.

3) There is a type of diagram which is allowed for case (i) with the physical Higgs field ϕ_h alone, but which is forbidden for case (ii) when we consider the complex doublet Higgs field ϕ_H which carries a charge of weak hypercharge. In fact there is for case (i), with only the physical Higgs field being considered, the possibility of "crossing" the two Higgs propagators in the box diagram. Because of this possibility, we get a factor of 2 bigger amplitude for the case (i). This means that going from case (i) to the four component case (ii), one gets a factor of $1/2$.

Altogether we thus get an increase by a factor of $\sqrt{2}^4 * 2 * \frac{1}{2} = 4$, by including all four components instead of just the physical Higgs, and that is just what we argued for

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[8]Note that when we formally only consider the left-handed quarks in a loop, it means that we have ignored the quark mass and left it as a perturbation to be considered later.

in Sec. 3.

Appendix F: Distribution of lengths of loops

In Sec. 4 we made the assumption that, without the weighting coming from the number of isodoublet states that can circle in a loop of n "propagators", the number of such loops statistically had a smooth distribution as a function of n, although there only are loops with an even number n.

We here want to consider this assumption in a little more detail: Imagine that we construct a random diagram by going along in small steps following the construction of a loop of propagators for the isodoublet particles [i.e., left-handed b or t quarks or Higgs particles (including the eaten Higgses)]. Then as one goes along it is sensible to think that, almost all the time, there is the same chance of getting back to the starting point of the initiated loop. This chance of getting back to the starting point should, namely, all the time be roughly 1 divided by the number of possible attachment points (say the order of the diagram) for an isodoublet propagator getting inserted. We must admit however that we have not clearly stated which way one should imagine to build up the diagram. One way would be to imagine that the structure of the diagram is already given and one just successively attaches a label, doublet or singlet, to the propagators in an already given diagram. One would still have to think of the given diagram statistically only and that the chance for the doublet loop being followed reaching any *a priori* vertex could be taken to be the same all through the construction. Then, although in principle the possible attachment points at any stage of the construction become all the vertices not yet used, we do not correct for this fact that vertices already used are no longer accessible.

This crude argumentation will give an exponentially decaying distribution for the distribution of the loop length n. However it will fall off so slowly with large n that the average loop length gets of the order of the full number of attachment possibilities. This means, assuming a large diagram, a very flat distribution for the first few n values to the extent that they can at all be realized. (For instance, $n = 2$ would only occur inside self-energy diagrams for the right-handed top quark, and also only even n are possible.)

Appendix G: Flattening of potential for small r

For the purpose of estimating an effective Higgs mass to take into account the difference between the Yukawa and Coulomb potentials, we want first to estimate how the Higgs field varies with the distance r from the center of the bound state. Strictly speaking we should calculate the wave function distribution for the constituents and evaluate the Higgs field

with this density of constituents used as the source. However, we shall here approximate the correct density distribution by a distribution, ρ_0, that is constant in 3-space inside a radius R, i.e., for $r < R$, and zero outside. Its value is chosen so as to correspond to there being a "charge" (i.e. the number of constituents times $g_t/\sqrt{2}$) of $(11/2)g_t/\sqrt{2}$. We shall take the R parameter then to be the average radius of the wave function distribution, i.e.,

$$R = <r> = \frac{3}{2}r_0. \tag{135}$$

In the very center there must (statistically) be a certain density of constituent particles having an extremum there. This means that, in the immediate neighborhood of the center, the density of constituents goes as

$$\rho(r) \approx \rho(0). \tag{136}$$

This leads to a spherically symmetric potential or Higgs field, satisfying the Laplace equation with the source term

$$g_t/\sqrt{2} * \rho = \rho_0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\phi_h}{\partial r}\right). \tag{137}$$

The resulting potential is

$$V = \frac{\rho_0}{6}r^2 + C \quad \text{for } r \leq R. \tag{138}$$

The physical number of constituents inside the average radius R is half the number of constituents in total and we thus identify it with our number of effective constituents in the center $Z = 11/2$. Thus we find

$$\rho_0 \approx \frac{Zg_t/\sqrt{2}}{4\pi R^3/3} \approx \frac{11/2 * g_t/\sqrt{2}}{4\pi/3* <r>^3}. \tag{139}$$

If we also use Z=11/2 for the outside field, being approximated as a Coulomb potential, we shall automatically get that the slope of the potential is continuous:

$$V = \begin{cases} \frac{\rho_0}{6}r^2 + C & \text{for } r \leq R \\ -\frac{Zg_t/\sqrt{2}}{4\pi r} & \text{for } r \geq R \end{cases} \tag{140}$$

Inserting (139) and adjusting C to make the two expressions coincide for $r = R$ leads to

$$V = \begin{cases} \frac{Zg_t/\sqrt{2}}{8\pi R^3}(r^2 - 3R^2) & \text{for } r \leq R \\ -\frac{Zg_t/\sqrt{2}}{4\pi r} & \text{for } r \geq R \end{cases} \tag{141}$$

$$= \frac{Zg_t/\sqrt{2}}{8\pi R^3}\begin{cases} r^2 - 3R^2 & \text{for } r \leq R \\ -\frac{2R^3}{r} & \text{for } r \geq R \end{cases} \tag{142}$$

It is easily seen that the variation of this potential, or the deviation of the Higgs field from the usual VEV, over the range r running from R to ∞ is twice that over the range

of r going from 0 to R. Thus the potential variation from $r \to \infty$ to $r = 0$ is $3/2$ times that from $r \to \infty$ to $r = R$ (which we took as the average radius).

As we saw in Sec. 5, for the case where gluons are ignored, the Higgs field became zero at the average radius, R. So, in this case, the central value of the Higgs field would be

$$\phi_h|_{r=0} = -\frac{1}{2} < \phi_h > = -\frac{1}{2}v \tag{143}$$

i.e., opposite in sign and half the magnitude of the usual VEV v. However, when we take into account the gluon part of the binding (see Sec. 5.1), we only need the potential at the average distance to be $4/9$ of what it was for the case of ignoring the gluons. In this case we got the Higgs field at the average distance $R = < r >$ to be $(1 - 4/9)v = 5v/9$. Then the field strength at the center becomes

$$\phi_h|_{r=0} = (1 - 3/2 * 4/9)v = v/3. \tag{144}$$

For $r \leq r_{inflection}$ the second derivative of the effective potential $V_{eff}(\phi_h)$ for the Higgs field is negative, so that the effective Higgs mass in this region is imaginary. We now want to get a typical average value for this second derivative to be used in estimating the effective imaginary Higgs mass in this range.

For orientation we note that, while the second derivative of $V_{eff}(\phi_h)$ at the inflection point where $r = r_{inflection}$ is by definition just zero, we have that the Higgs field takes the value $v/3$ at $r = 0$ when gluons are included. Now, for the Higgs field $\phi_h = v/3$, the second derivative of the V_{eff} is $-\frac{1}{3}$ times its value at the minimum of the effective potential V_{eff}, i.e., where it is equal to the physical Higgs mass squared. Thus the effective Higgs mass squared at the value of the field in the central region of the bound state is $-\frac{1}{3}m_h^2$, where m_h is the physical Higgs mass.

In order to get an estimate of the effective imaginary Higgs mass in the region of r going from 0 to $r_{inflection}$, we may linearly interpolate the second derivative as a function of r but then remember to weight the importance of the various r regions with the weight factor r^2. The first step in our crude estimate is to approximate the second derivative as a linear function in the distance from the center r,

$$\frac{d^2V(r)}{d\phi_h^2} = -\frac{1}{3}m_h^2\left(1 - \frac{r}{r_{inflection}}\right). \tag{145}$$

Introducing the notation $x = \frac{r}{r_{inflection}}$, we then see that the average value of $\frac{d^2V(r)}{d\phi_h^2}$, weighted with r^2, in the range $r \in [0, r_{inflection}]$ is

$$< \frac{d^2V(r)}{d\phi_h^2} >= -\frac{1}{3}m_h^2\frac{\int_0^1 x^2(1-x)dx}{\int_0^1 x^2dx} = -\frac{m_h^2}{12}. \tag{146}$$

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It is easily seen that indeed the effective mass squared $m_{h\ eff}^2$ of the Higgs is just

$$m_{h\ eff}^2 = \frac{d^2 V(r)}{d\phi_h^2}. \tag{147}$$

So that for its average value we get

$$m_{h\ eff}^2 = -\frac{m_h^2}{12}. \tag{148}$$

Appendix H, Bound state mass dependence on the number of constituents

In Sec. 6.5 we need an estimate of the mass of the 10-constituent bound state rather than that of the 12-constituent bound state, which we are requiring to be massless. We shall therefore present here a first estimate of the form of the dependence of the mass squared of our family of bound states on the number of (t and $\bar{t}$) constituents $\hat{Z} = Z + 1$.

We argued in Appendix B that the mass squared $M^2 = m_{bound}^2$ of the bound state should be an analytic function of the "parameters", such as g_t or even, as we shall use here, of $\hat{Z}$. In other words we shall assume that the mass squared of the bound state $M^2(\hat{Z})$ is an analytic function of the number of constituents $\hat{Z}$.

In the weak coupling approximation (i.e., g_t and α_s small), the mass of the bound state becomes $M(\hat{Z}) \approx m_t \hat{Z}$, since it is essentially given by adding the masses of the constituents. This is a reasonable approximation for small $\hat{Z}$ and thus we obtain

$$M^2(\hat{Z}) \approx m_t^2 \hat{Z}^2 \tag{149}$$

as a valid approximation for small $\hat{Z}$.

Now, however, there is a binding energy term, which becomes bigger and bigger as $\hat{Z}$ increases. The total potential energy of the constituents is proportional to the number of interacting pairs and is thus proportional to $\hat{Z}^2$ or strictly speaking $\hat{Z}(\hat{Z} - 1)$. Hence each constituent feels a potential proportional to $\hat{Z}^2/\hat{Z} = \hat{Z}$ or strictly $\hat{Z}(\hat{Z} - 1)/\hat{Z} = \hat{Z} - 1$. At the same time the average distance of the constituent from the center of the bound state is diminished, as in the hydrogen atom, in the same proportion. It follows that the binding energy per particle becomes proportional to the square of this factor. So the total binding energy of the bound state is proportional to $\hat{Z}\hat{Z}^2$ or strictly $\hat{Z}(\hat{Z} - 1)^2$.

Thus we are led to the following Taylor expansion of $M^2(\hat{Z})$:

$$M^2(\hat{Z}) = (m_t \hat{Z} - A\hat{Z}^3 + \cdots)^2 = m_t^2 \hat{Z}^2 (1 - B\hat{Z}^2 + \cdots) \tag{150}$$

or strictly speaking

$$M^2(\hat{Z}) = (m_t \hat{Z} - A'\hat{Z}(\hat{Z} - 1)^2 + \cdots)^2 = m_t^2 \hat{Z}^2 (1 - B'(\hat{Z} - 1)^2 + \cdots) \tag{151}$$

The main point of the present article is to investigate the hypothesis that the top-quark Yukawa coupling is fine-tuned, so as to make the mass of the bound state with $\hat{Z} = 12$ constituents just zero. Imposing this requirement onto the above Taylor expansion leads to a smooth ansatz of the form

$$M^2(\hat{Z}) = m_t^2 \hat{Z}^2 \left(1 - \left(\frac{\hat{Z}}{12} \right)^2 \right) \tag{152}$$

or strictly speaking we should have

$$M^2(\hat{Z}) = m_t^2 \hat{Z}^2 \left(1 - \left(\frac{(\hat{Z}-1)}{11} \right)^2 \right). \tag{153}$$

We now use the Taylor expansion (152) to give a first order estimate of the masses for the 11- and 10-constituent bound states:

$$m_{11} = \sqrt{11^2 m_t^2 \left(1 - \left(\frac{11}{12} \right)^2 \right)} = 4.4 m_t = 760 \text{ GeV} \tag{154}$$

while

$$m_{10} = \sqrt{10^2 m_t^2 \left(1 - \left(\frac{10}{12} \right)^2 \right)} = 5.5 m_t = 950 \text{ GeV} \tag{155}$$
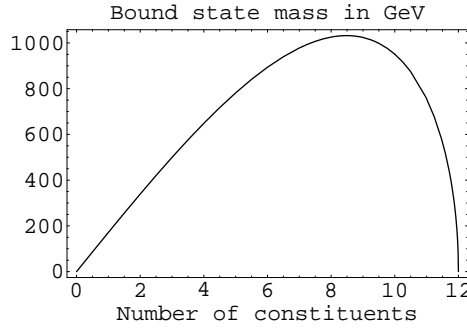
The full spectrum is shown in Fig. 1.

Figure 1: Mass spectrum of bound states.

Appendix I: Gaussian wave function ansatz

We shall now construct an ansatz for an approximation to the multiparticle wave function for our system, consisting of the 6 top and 6 antitop particles, based on Gaussian functions. The main purpose of this exercise is to confirm from a concrete model ansatz the major part, namely, a factor 2 in the binding energy, of the many body correction of Sec. 9.

The ansatz wave function for the N-particle system proposed here is simply of the form

$$\psi(\vec{x}_1, \cdots, \vec{x}_N) = \mathcal{N} \prod_{i=1}^{N} \exp(-a_i \vec{x}_i^2). \tag{156}$$

Of course, in our case of 12 constituents, we have $N = 12$. The idea then is to use the Hamiltonian based on the application of the potential V_{total} from Eq. (8) and the kinetic energy summed over the N particles, or we may simply use $H = \sum_i H_i$ with H_i taken from Eq. (22):

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m_i} + \frac{1}{2} \sum_{i,j,\ i \neq j} V_{ij}. \tag{157}$$

Here $V_{ij} = \frac{A}{4\pi r_{ij}}$ is given by Eq. (23), with $r_{ij} = |\vec{x}_i - \vec{x}_j|$ being the distance between particle number i and particle number j.

The idea now is that we imagine to find the best possible wave function of this form for the bound state system, by evaluating the average energy in such an ansatz state as a function of the parameters a_i. In practice, for our symmetric case, we obtain the same value for all the N a_i's and simply minimize the energy with respect to their common value $a_i = a$. Using our Gaussian ansatz, we obtain

$$< H_i >= \frac{3a}{2m_t} - (N-1)\frac{A}{4\pi}\sqrt{\frac{a}{\pi}} \tag{158}$$

for the expectation value of the single particle Hamiltonian H_i. Minimizing this energy determines our variational parameter to be

$$a = (N-1)^2 \left(\frac{A}{4\pi}\right)^2 \frac{m_t^2}{9\pi}. \tag{159}$$

This gives, using

$$< H >= N < H_i >= -\frac{N(N-1)^2 A^2 m_t}{96\pi^3} = -\frac{(e_{t\bar{t}}^2 + 4g_t^2)^2}{32\pi^2}\frac{4}{3\pi}m_t \tag{160}$$

for the factorizable Gaussian wave function estimate of (minus) the binding energy of the bound state, where we have substituted $N = 12$ and the expression for A from Eq. (23).

We can now compare this value (160) for the binding energy[9] with the "Bohr model" approximation $E_{binding}$ of Eq. (10):

$$- < H >= E_{binding} * \frac{1}{2} * \frac{12}{11} * \frac{8}{3\pi} = \frac{E_{binding}}{2.16}. \tag{161}$$

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[9]We note that this value is in agreement with the calculation of the many body effect in a recent paper [5] by Kuchiev, Flambaum and Shuryak, when the correction by a factor of 2 mentioned in footnote 1 and the reduced mass factor of $11/12$ are taken into account. In fact it means that in the notation of Ref. [5] we would obtain $k = 1/6\pi \approx 0.053$, while in their variational calaculation they obtain $k = 25/512 \approx 0.049$.

Thus we have basically reproduced the expected main reduction in the binding energy by a factor of 2 due to many body effects. We note that the extra factor of $\frac{11}{12}$ arises from the reduced mass of a single quark moving relative to the other 11 quarks after removing the center of mass motion. The final factor of $\frac{8}{3\pi}$ corresponds to the reduction in the Bohr model binding energy obtained by using a Gaussian form rather than the exact Bohr wave function.

Appendix J: Phase transition in bound state calculation

We shall illustrate the possibility for the appearance of a phase transition in the bound state calculation, which can explain the disagreement between the present paper and Ref. [5]. For reasons of tractability we do not consider the genuine bound state calculation, but rather a toy model that simulates a continuous material made from such bound states and extended to infinity.

Really our toy model is a material with an *a priori* fixed density of both top and antitop quarks. But then the idea is to adjust the density of top and antitop quarks so as to correspond to the situation in which the bound states just fill the space completely without overlapping.

It is important that we treat top and antitop quarks as different species of the same type of particle, which are separately conserved. So it only matters how many top or antitop quarks there are together in states with a given momentum. The number of possible states for a given momentum is denoted by $N_{sp} = 2 * 2 * N_c$. Here N_c is the number of colors. So $N_{sp} = 12$ is the case of interest for nature and is the value we use below. As part of our toy model we ignore annihilation completely, so that particles and antiparticles are separately conserved. Then we take the Fermi momentum p_f as an ansatz parameter. From this alone we can derive the density of the top quarks and antitop quarks in the ansatz state:

$$\rho = N_{sp} \frac{4\pi p_f^3}{3(2\pi)^3}. \tag{162}$$

Their energy density is then

$$\text{``energy density of fermions''} = \frac{N_{sp}}{(2\pi)^3} \int_0^{p_f} 4\pi p^2 \sqrt{p^2 + m^2} dp. \tag{163}$$

Now the fermion mass comes from the Higgs field and we have

$$m = g_t < \phi_h > /\sqrt{2}. \tag{164}$$

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The potential energy for the Higgs field ϕ_h is of the form

$$V_{eff}(\phi_h) = -\frac{1}{2}|m_{hb}|^2\phi_h^2 + \frac{\lambda}{8}\phi_h^4. \tag{165}$$

For use in the present Appendix, we introduce the effective potential normalized to be just zero at the (usual) minimum:

$$\begin{aligned}
V_{effnorm}(\phi_h) &= -\frac{1}{2}|m_{hb}|^2\phi_h^2 + \frac{\lambda}{8}\phi_h^4 - V_{eff}(<\phi_h>) \tag{166} \\
&= -\frac{1}{2}|m_{hb}|^2\phi_h^2 + \frac{\lambda}{8}\phi_h^4 + \frac{|m_{hb}|^4}{2\lambda}. \tag{167}
\end{aligned}$$

We consider the approximation in which the Higgs field ϕ_h is taken to have a constant value inside the bound state. So the kinetic energy of the Higgs field can be ignored and thus the total energy density U in our toy model ansatz becomes

$$\begin{aligned}
U &= \frac{N_{sp}}{2\pi^2}\int_0^{p_f} p^2\sqrt{p^2 + (g_t\phi_h/\sqrt{2})^2}dp + V_{effnorm}(\phi_h) \tag{168} \\
&= \frac{N_{sp}}{32\pi^2}\left(\frac{(g_t\phi_h)^4}{4}\left[\log\frac{(g_t\phi_h)^2}{2} - 2\log(p_f + \sqrt{p_f^2 + (g_t\phi_h)^2/2})\right]\right) \\
&\quad + \frac{N_{sp}}{32\pi^2}\left(2p_f(2p_f^2 + \frac{(g_t\phi_h)^2}{2})\sqrt{p_f^2 + (g_t\phi_h)^2/2}\right) + V_{effnorm}(\phi_h). \tag{169}
\end{aligned}$$

The Fermi momentum p_f really determines the density of quarks or antiquarks and thus—if bound states are effectively present—also the density of the bound states. Now we want to adjust the density in such a way as to crudely represent the fact that the space is filled up with bound states, so that in every point of space there is just one of the bound states present. That is to say we must adjust the Fermi-momentum p_f to such a value that we achieve this density corresponding to totally filling space with bound states. After adjusting p_f in such a way, we can obtain the mass or rather the energy of the bound state by using the fact that the number of bound states per unit volume is

$$\frac{4\pi}{3(2\pi)^3}p_f^3 = \frac{p_f^3}{6\pi^2}. \tag{170}$$

So the mass or rather the energy of the potential bound state (ENBS) is

$$ENBS = \frac{6\pi^2}{p_f^3}U. \tag{171}$$

Now the density needed to have filled space with the bound states can—crudely at least—be found by minimizing the bound state energy ENBS with respect to the variable determining the density, i.e., with respect to p_f. The argument for this runs as follows:

1) If we make an ansatz "material" with a lower density of bound states than there is place for, then each bound state can be imagined to be surrounded by a little piece of

essentially vacuum the energy of which must be added to the value ENBS as calculated from (171). Now we have normalized the effective potential $V_{effnorm}(\phi_h)$ by making it vanish at its minimum. So the pieces of new vacuum in any ansatz will have positive energy. Thus, if we make the density in the ansatz too low, the result for ENBS will always be larger than the true bound state energy.

2) On the other hand if we make an ansatz with a too high density so that the bound states get squeezed together, this will also cause the energy per bound state ENBS to increase compared to that of a free bound state.

So we see that the energy formally calculated from an ansatz ENBS will be bigger both when the density is higher and when it is lower than the one corresponding to the bound states just touching or filling the space. This then means that there must be a minimum in the energy per bound state ENBS as a function of the density parameter p_f.

Since we are working in the approximation of letting the Higgs field be constant inside the bound state, we really just want to adjust this Higgs field ϕ_h so as to minimize the energy of the bound state. Combined with the above-mentioned adjustment of p_f, we end up with the rule that we shall adjust both parameters p_f and ϕ_h so as to minimize the expression (171) for ENBS. Then we should obtain, in our ansatz approximation, the right mass or rather energy for the bound state if there is a bound state. If there is no binding, we should get the energy of the N_{sp} "constituents" that were meant to be bound. In the case of a potential bound state made from $N_{sp} = 12$ top or antitop quarks, this constituent energy would of course be N_{sp} times the top-quark mass (or energy, but we expect that the speed would be low in our ansatz).

The main point of this Appendix is that *the mass or energy of the bound state appears as the result of taking a minimum so that it will not normally be a nice analytical function of the parameters that are input into the calculation such as g_t, but rather tends to have a kink as a function of the inputs.*

Without taking into account on which side of the "phase transition" a given value of g_t may lie, one can *a priori* make a severe error in the calculation. According to our toy model, the correct side of the phase transition is determined by the question of whether or not the Higgs field in the region of the potential bound state has been pushed so much as to deviate strongly from its value in the usual vacuum. The calculation in Ref. [5] has been made on the small g_t side of the phase transition, where to a very good approximation we have the usual vacuum with the usual 246 GeV Higgs field expectation value. On the other hand, in this paper we have worked in the regime where we take the Higgs field in the interior of the hypothesized bound state to deviate significantly from that in the usual vacuum. Indeed the typical field value inside the bound state in our calculation is rather small. So we have worked on the large g_t side of the phase transition.

We now present the results of our toy model calculation, which exhibit the existence of such a phase transition. Here we use a Higgs mass of $m_h = 115$ GeV. The results obtained for the mass or really the energy of the potential bound state ENBS are plotted in Fig. 2 as a function of g_t.
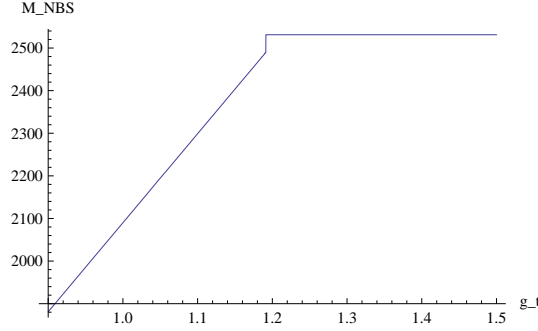

Figure 2: Energy or mass of the "bound state" in GeV in our toy model as a function of the top-quark Yukawa coupling g_t.

They were calculated by simply minimizing, for each choice of the Yukawa coupling g_t, the value of ENBS as given by (171) with respect to both variables, the Fermi momentum p_f and the Higgs field (in the interior of the bound state) ϕ_h. The little step in the figure is an artifact of the calculational accuracy, but the kink is of course due to the minimum giving the smallest ENBS jumping discontinuously at $g_t = 1.191$. Indeed the minimum jumps from $(\phi_h, p_f) = (246, 0.18$ GeV$)$ to $(0, 211$ GeV$)$.

This jumping is partly illustrated by Fig. 3, where the potential bound state mass or rather energy ENBS is plotted as a function of ϕ_h, when the latter is imposed as the approximate value of the Higgs field inside the bound state region. It means that for every ϕ_h value the function ENBS from (171) has been minimized with respect to effectively the density of bound states, meaning minimization with respect to p_f. Figure 3 is made for the specific value $g_t = 1.191$, which is the phase transition value. This is reflected by the fact that you see two essentially degenerate minima in Fig. 3.

For g_t greater than the phase transition value of 1.191, the mass or energy remains constant as the Yukawa coupling g_t increases. This means that the binding gets stronger and stronger, in as far as the binding energy is really

$$\text{"binding"} = \frac{N_{sp}g_t}{\sqrt{2}}246 \text{ GeV} - ENBS. \tag{172}$$

Thus, for example, in our toy model the binding energy becomes equal to half the mass of the constituents for $g_t = 2.42$. According to our discussion in Sec. 2.4, this is the formal requirement for a massless bound state. Thus, in the bad approximation of ignoring the exchange of eaten Higgses, gluon exchange etc. and even *taking the Higgs field inside the*
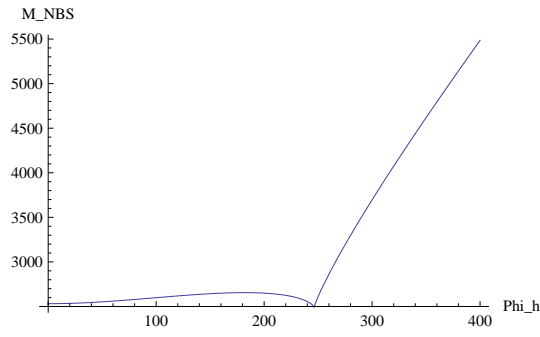
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Figure 3: Energy or mass of the bound state in GeV in our toy model as a function of the imposed ϕ_h value, but with p_f adjusted by minimization. The top-quark Yukawa coupling is chosen to have the phase transition value $g_t = 1.191$.

bound state as constant, we obtain $g_t = 2.42$ as the value of the Yukawa coupling which gives a massless bound state in our toy model.

For the case of g_t less than the phase transition value of 1.191, we get a very small value for p_f compared to our own results from the Bohr atom approximation. We get $p_f \sim 0.18$ GeV rather than of order $g_t^2 m_t$. This very small value of p_f may be interpreted as supporting (as does Fig. 2 for $g_t < 1.191$) the result of Kuchiev et al. according to which the system does not bind. Completely zero binding would correspond to each particle standing still and well separated from each other, which would imply a very low density and thus correspond to $p_f = 0$.

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